Introduction to Nonlinear Control

Stability, control design, and estimation

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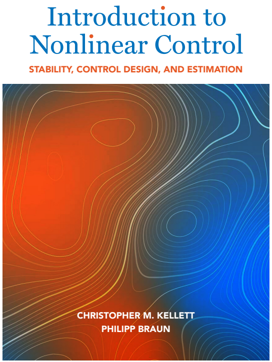

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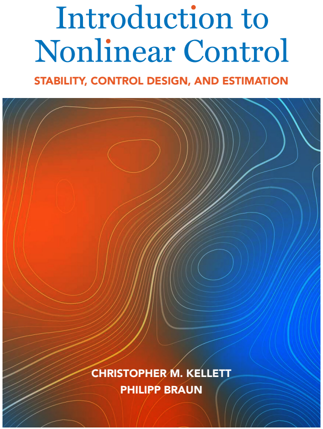
Part I: Dynamical Systems



Stability Notions

Consider: Autonomous system

$$\dot{x} = f(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

Question: What can be said about $x(t)$ for $t \to \infty$?

Recall:

- $x^e \in \mathbb{R}^n$ is called an equilibrium if

$$\dot{x} = f(x^e) = 0$$

- Observe that:

$$x(t) = x^e \quad \forall t \in \mathbb{R}_{\geq 0} \text{ if } x(0) = x^e$$

(a special solution of the autonomous system)

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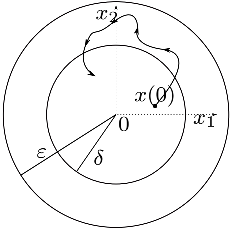

Definition (Stability Notations)

Consider $\dot{x} = f(x)$ with $f(0) = 0$.

- The origin is *(Lyapunov) stable* if, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$|x(0)| \leq \delta \quad \text{implies} \quad |x(t)| \leq \varepsilon \quad \forall\, t \geq 0.$$

- The origin is *unstable* if it is not stable.
- The origin is *attractive* if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{t\to\infty} x(t) = 0.$$

- The origin is *asymptotically stable* for $\dot{x} = f(x)$ if it is both stable and attractive.

Stability Notions: Simple Examples

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Consider the simple autonomous systems (with equilibrium at the origin):

$$\dot{x} = 0, \qquad \rightsquigarrow \qquad x(t) = x_0 \tag{1}$$

$$\dot{x} = x, \qquad \rightsquigarrow \qquad x(t) = e^t x_0 \tag{2}$$

$$\dot{x} = -x \qquad \rightsquigarrow \qquad x(t) = e^{-t} x_0 \tag{3}$$

This implies that: The origin of

- (1) is stable (but not attractive)
- (2) is unstable
- (3) is stable and attractive (i.e., asymptotically stable)

How to verify stability properties without knowledge of solutions?

Consider:

$$\dot{x} = f(x) \quad \text{with} \quad f(0) = 0, \quad x \in \mathbb{R}^n$$

Theorem (Lyapunov stability theorem)

Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \longleftarrow \text{ (Technical condition)}$$

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x) \rangle \leq 0 \quad \longleftarrow \text{ (Decrease condition)}$$

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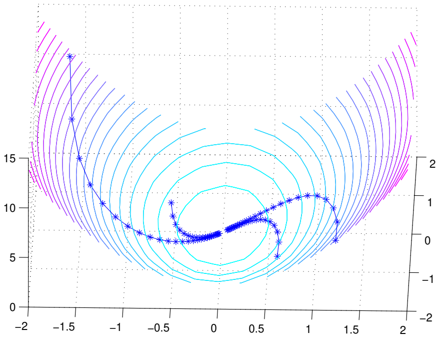
Interpretation of the decrease condition

(Forward invariance of sublevel sets)



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Observations & Extensions:

- Time derivative of the "generalized energy function" V along solutions:

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Simple exercise: Use $V(x) = x^2$ to show that the origin of

- $\dot{x} = 0$ is stable
- $\dot{x} = -x$ is asymptotically stable

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