

Introduction to Nonlinear Control

Stability, control design, and estimation

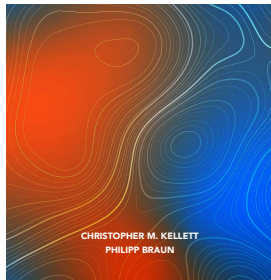
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

CHRISTOPHER M. KELLETT
PHILIPP BRAUN



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2.1.2 Time-Varying Systems

2.2 Comparison Principle

2.3 Stability by Lyapunov's Second Method

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2.3.2 Instability

2.3.2 Partial Convergence and the LaSalle-Yoshizawa Theorem

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2.6.1 Krasovskii-LaSalle Invariance Theorem

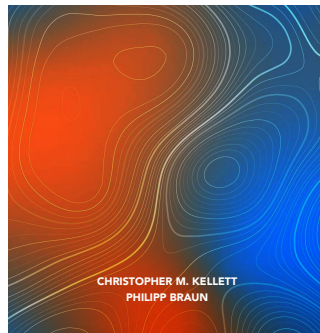
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Stability Notions

Consider: Autonomous system

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n$$

Question: What can be said about $x(t)$ for $t \rightarrow \infty$?

Recall:

- $x^e \in \mathbb{R}^n$ is called an equilibrium if

$$\dot{x} = f(x^e) = 0$$

- Observe that:

$$x(t) = x^e \quad \forall t \in \mathbb{R}_{\geq 0} \quad \text{if } x(0) = x^e$$

(a special solution of the autonomous system)

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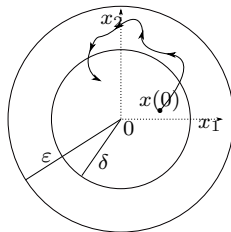
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Definition (Stability Notations)

Consider $\dot{x} = f(x)$ with $f(0) = 0$.

- The origin is **(Lyapunov) stable** if, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$|x(0)| \leq \delta \quad \text{implies} \quad |x(t)| \leq \varepsilon \quad \forall t \geq 0.$$

- The origin is **unstable** if it is not stable.
- The origin is **attractive** if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

- The origin is **asymptotically stable** for $\dot{x} = f(x)$ if it is both **stable** and **attractive**.

Stability Notions: Simple Examples

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- The origin is *asymptotically stable* for $\dot{x} = f(x)$ if it is both *stable* and *attractive*.

Consider the simple autonomous systems (with equilibrium at the origin):

$$\dot{x} = 0, \quad \rightsquigarrow \quad x(t) = x_0 \quad (1)$$

$$\dot{x} = x, \quad \rightsquigarrow \quad x(t) = e^t x_0 \quad (2)$$

$$\dot{x} = -x \quad \rightsquigarrow \quad x(t) = e^{-t} x_0 \quad (3)$$

This implies that: The origin of

- (1) is stable (but not attractive)
- (2) is unstable
- (3) is stable and attractive (i.e., asymptotically stable)

How to verify stability properties without knowledge of solutions?

Consider:

$$\dot{x} = f(x) \quad \text{with} \quad f(0) = 0, \quad x \in \mathbb{R}^n$$

Theorem (Lyapunov stability theorem)

Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \longleftarrow \text{(Technical condition)}$$

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x) \rangle \leq 0 \quad \longleftarrow \text{(Decrease condition)}$$

Then the origin is (globally) stable.

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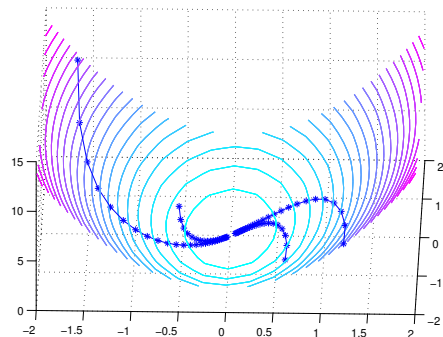
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Interpretation of the decrease condition

(Forward invariance of sublevel sets)



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Observations & Extensions:

- Time derivative of the “generalized energy function” V along solutions:

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Simple exercise: Use $V(x) = x^2$ to show that the origin of

- $\dot{x} = 0$ is stable
- $\dot{x} = -x$ is asymptotically stable

Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
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4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

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9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

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17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems