Introduction to Nonlinear Control

Stability, control design, and estimation

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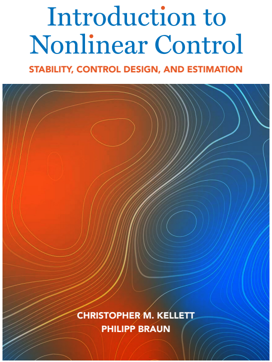
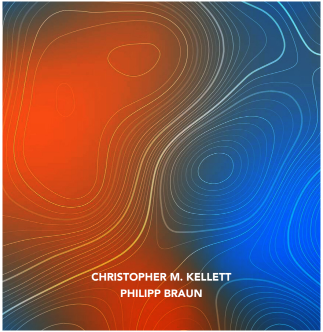


Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



What is a linear system and why do we care about linear systems?

General (time-invariant) system:

$$\dot{x} = f(x), \qquad f : \mathbb{R}^n \to \mathbb{R}^n$$

Linear (time-invariant) system:

$$\dot{x} = Ax, \qquad A \in \mathbb{R}^{n \times n}$$

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- We know how the solution looks like (namely $x(t) = e^{At}x_0$)
- The origin $x = 0$ is always an equilibrium
- The eigenvalues of A completely determine the stability properties of the origin.
- local stability properties are equivalent to global stability properties
- asymptotic stability is equivalent to exponential stability

Theorem (Stability of linear systems)

For the linear system $\dot{x} = Ax$, the origin is

1. *stable if and only if the eigenvalues of A have negative or zero real parts and all Jordan blocks corresponding to eigenvalues with zero real parts are 1×1;*
2. *unstable if and only if at least one eigenvalue of A has a positive real part or zero real part with the corresponding Jordan block larger than 1×1;*
3. *exponentially stable if and only if all the eigenvalues of A have strictly negative real parts.*

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- asymptotic stability is equivalent to exponential stability
- stability can be characterized through quadratic Lyapunov functions: $V(x) = x^T Px$

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Theorem

For the linear system $\dot{x} = Ax$, the following are equivalent:

1. *The origin is exponentially stable;*
2. *All eigenvalues of A have strictly negative real parts;*
3. *For every $Q \in \mathcal{S}^n_{>0}$ there exists a unique $P \in \mathcal{S}^n_{>0}$, satisfying the Lyapunov equation*

$$A^T P + PA = -Q.$$

Linearization (Local exponential stability)

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Define linearization of $\dot{x} = f(x)$ at $x = 0$:

$$\dot{z}(t) = Az(t)$$

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Consider $\dot{x} = f(x)$ (f cont. differentiable) and its linearization $\dot{z} = Az$. If the origin $z^e = 0$ of $\dot{z} = Az$ is globally exponentially stable then the origin $x^e = 0$ of $\dot{x} = f(x)$ is locally exponentially stable.

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Discussion:

- Local exponential stability properties are preserved under linearization.
- In many cases, the linear approximation of a nonlinear system can be used to draw local conclusions about the nonlinear system.
- However, this is not always the case!

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Exercise:

- Consider $\dot{x} = -x^3$
  - Show that the origin of the nonlinear system is asymptotically stable.
  - Show that the origin of the linearization at $x = 0$ is not asymptotically stable.
- Consider $\dot{x} = x^3$
  - Show that the origin of the nonlinear system is unstable.
  - Show that the origin of the linearization at $x = 0$ is not unstable.

Linearization of dynamical systems with inputs and outputs

Consider:

$$\dot{x} = f(x,u), \quad (f\text{, cont. differentiable w.r.t. } x \text{ and } u)$$
$$y = h(x,u), \quad (h\text{, cont. differentiable w.r.t. } x \text{ and } u)$$

Recall:

- An equilibrium pair (x^e, u^e) satisfies $f(x^e, u^e) = 0$
- Without loss of generality $f(0,0) = 0$ and $h(0,0) = 0$ (due to coordinate transformation)

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Linear system with input and output:

$$\dot{x} = Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$$
$$y = Cx + Du, \qquad C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$$

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Note that

- A linear system (with output) is unambiguously defined through (A,B,C,D)
- (A,B) describes the system without output
- (A,C) describes output behavior without input
- The matrix D (direct feedthrough) is often not present

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For a linear system (A,B,C,D) we can define:

- controllability (i.e., ability to steer the state from an arbitrary $x_0 \in \mathbb{R}^n$ to an arbitrary x_1 through an appropriate input selection $u(t)$)
- observability (i.e., ability to reconstruct the state $x(t)$ by observing the output $y(t)$)

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Controller design:

- For a controllable system, pole placement can be used to define a stabilizing feedback law $u = Kx$ (i.e., the origin of $\dot{x} = (A+BK)x$ is asymptotically stable)

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> In many cases, locally, the analysis of linear systems is sufficient to draw conclusions about the corresponding nonlinear dynamics.

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