

Introduction to Nonlinear Control

Stability, control design, and estimation

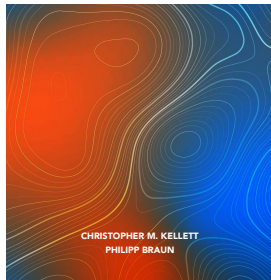
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

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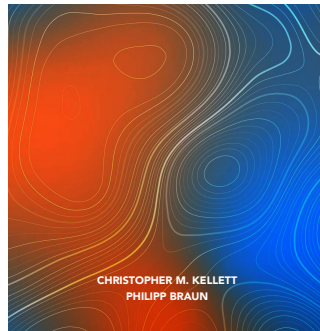
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Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



What is a linear system and why do we care about linear systems?

General (time-invariant) system:

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Linear (time-invariant) system:

$$\dot{x} = Ax, \quad A \in \mathbb{R}^{n \times n}$$

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Properties of linear systems:

- We know how the solution looks like (namely $x(t) = e^{At}x_0$)
- The origin $x = 0$ is always an equilibrium
- The eigenvalues of A completely determine the stability properties of the origin.
- local stability properties are equivalent to global stability properties
- asymptotic stability is equivalent to exponential stability

Theorem (Stability of linear systems)

For the linear system $\dot{x} = Ax$, the origin is

- 1 **stable if and only if** the eigenvalues of A have negative or zero real parts and all Jordan blocks corresponding to eigenvalues with zero real parts are 1×1 ;
- 2 **unstable if and only if** at least one eigenvalue of A has a positive real part or zero real part with the corresponding Jordan block larger than 1×1 ;
- 3 **exponentially stable if and only if** all the eigenvalues of A have strictly negative real parts.

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- asymptotic stability is equivalent to exponential stability
- stability can be characterized through quadratic Lyapunov functions: $V(x) = x^T P x$

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Theorem

For the linear system $\dot{x} = Ax$, the following are **equivalent**:

- 1 The origin is **exponentially stable**;
- 2 All eigenvalues of A have strictly negative real parts;
- 3 For every $Q \in S_{>0}^n$ there exists a unique $P \in S_{>0}^n$, satisfying the **Lyapunov equation**

$$A^T P + P A = -Q.$$

Linearization (Local exponential stability)

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Consider $\dot{x} = f(x)$ (f cont. differentiable) and its linearization $\dot{z} = Az$. **If** the origin $z^e = 0$ of $\dot{z} = Az$ is **globally exponentially stable** **then** the origin $x^e = 0$ of $\dot{x} = f(x)$ is **locally exponentially stable**.

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Discussion:

- **Local** exponential stability properties are preserved under linearization.
- In many cases, the linear approximation of a nonlinear system can be used to draw local conclusions about the nonlinear system.
- However, this is not always the case!

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Exercise:

- Consider $\dot{x} = -x^3$
 - ▶ Show that the origin of the nonlinear system is asymptotically stable.
 - ▶ Show that the origin of the linearization at $x = 0$ is not asymptotically stable.
- Consider $\dot{x} = x^3$
 - ▶ Show that the origin of the nonlinear system is unstable.
 - ▶ Show that the origin of the linearization at $x = 0$ is not unstable.

Linearization of dynamical systems with inputs and outputs

Consider:

$$\dot{x} = f(x, u), \quad (f, \text{cont. differentiable w.r.t. } x \text{ and } u)$$

$$y = h(x, u), \quad (h, \text{cont. differentiable w.r.t. } x \text{ and } u)$$

Recall:

- An equilibrium pair (x^e, u^e) satisfies $f(x^e, u^e) = 0$
- Without loss of generality $f(0, 0) = 0$ and $h(0, 0) = 0$
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Linear system with input and output:

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$$

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Note that

- A linear system (with output) is unambiguously defined through (A, B, C, D)
- (A, B) describes the system without output
- (A, C) describes output behavior without input
- The matrix D (direct feedthrough) is often not present

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For a linear system (A, B, C, D) we can define:

- **controllability** (i.e., ability to steer the state from an arbitrary $x_0 \in \mathbb{R}^n$ to an arbitrary x_1 through an appropriate input selection $u(t)$)
- **observability** (i.e., ability to reconstruct the state $x(t)$ by observing the output $y(t)$)

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Controller design:

- For a controllable system, **pole placement** can be used to define a stabilizing feedback law $u = Kx$ (i.e., the origin of $\dot{x} = (A + BK)x$ is asymptotically stable)

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In many cases, locally, the analysis of linear systems is sufficient to draw conclusions about the corresponding nonlinear dynamics.

Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
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4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
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17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems