

Introduction to Nonlinear Control

Stability, control design, and estimation

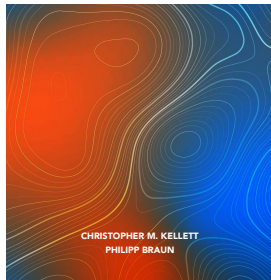
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

CHRISTOPHER M. KELLETT
PHILIPP BRAUN



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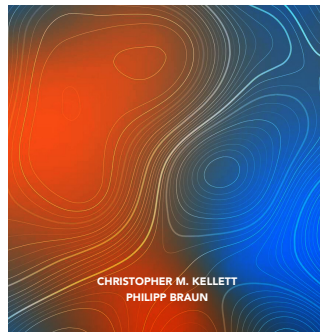
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Notation and assumptions:

- **Transfer function** $G : \mathbb{C} \rightarrow \mathbb{C}$.
- G is a rational function, i.e., there exist polynomial functions $P, Q \in \mathbb{R}[s]$ (with coefficients in \mathbb{R}) such that

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Definition (Inverse Laplace transform)

Consider $\hat{\varphi} : \mathcal{C} \rightarrow \mathbb{C}^m$ and let $\alpha \in \mathbb{R}$ such that $\alpha + j\beta \in \mathcal{C} \subset \mathbb{C}$ for all $\beta \in \mathbb{R}$. Then the inverse Laplace transform $\varphi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ of $\hat{\varphi}$ is defined as

$$\begin{aligned} \varphi(t) &= (\mathcal{L}^{-1}\hat{\varphi})(t) = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} e^{st} \hat{\varphi}(s) ds \\ &= \frac{e^{\alpha t}}{2\pi j} \int_{-\infty}^{\infty} e^{j\omega t} \hat{\varphi}(\alpha + j\omega) d\omega. \end{aligned}$$

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Proposition (Laplace transform properties)

Consider the **signals**

$$\varphi, \varphi_1, \varphi_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$$

in the time domain and **constants**

$$a \in \mathbb{R}_{>0}, \quad a_1, a_2 \in \mathbb{R}.$$

Then the **Laplace transform** satisfies

$$\mathcal{L}^{-1} \mathcal{L} \varphi(t) = \varphi(t),$$

$$\mathcal{L}(a_1 \varphi_1 + a_2 \varphi_2)(s) = a_1 \hat{\varphi}_1(s) + a_2 \hat{\varphi}_2(s),$$

$$\mathcal{L}(\varphi(a \cdot))(s) = \frac{1}{a} \hat{\varphi}\left(\frac{s}{a}\right),$$

$$\mathcal{L}(\varphi(\cdot - a))(s) = e^{-sa} \hat{\varphi}(s),$$

$$\mathcal{L}\left(\frac{d^k}{dt^k} \varphi\right)(s) = s^k \hat{\varphi}(s) - \sum_{j=1}^{k-1} s^{j-1} \frac{d^{k-1-j}}{dt^{k-1-j}} \varphi(0),$$

$$\mathcal{L}\left(\int_0^\cdot \varphi(\tau) d\tau\right)(s) = \frac{1}{s} \hat{\varphi}(s).$$

From the time domain to the frequency domain (and back)

Consider single-input single-output (SISO) linear systems:

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Application of the Laplace transform:

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + b\hat{u}(s), \quad \hat{y}(s) = c\hat{x}(s) + d\hat{u}(s)$$

Rearrange the terms ($x(0) = 0$):

$$\hat{y}(s) = (c(sI - A)^{-1}b + d) \hat{u}(s)$$

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Identify input output relationship:

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = c(sI - A)^{-1}b + d \quad (1)$$

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Definition (Realization)

Consider a transfer function $G(s)$ and assume that (1) is satisfied for (A, b, c, d) . Then $G(s)$ is called realizable and the quadruple (A, b, c, d) is called a realization of $G(s)$.

Proposition (Laplace transform properties)

Consider the signals $\varphi, \varphi_1, \varphi_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ in the time domain and constants $a \in \mathbb{R}_{>0}$, $a_1, a_2 \in \mathbb{R}$. Then the Laplace transform and its inverse satisfy the following properties:

$$\mathcal{L}^{-1} \mathcal{L} \varphi(t) = \varphi(t),$$

$$\mathcal{L}(a_1 \varphi + a_2 \varphi_2)(s) = a_1 \hat{\varphi}_1(s) + a_2 \hat{\varphi}_2(s),$$

$$\mathcal{L}(\varphi(a \cdot))(s) = \frac{1}{a} \hat{\varphi}\left(\frac{s}{a}\right),$$

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$$\mathcal{L}\left(\int_0^\cdot \varphi(\tau) d\tau\right)(s) = \frac{1}{s} \hat{\varphi}(s).$$

System Interconnections in the Frequency Domain

Consider two systems:

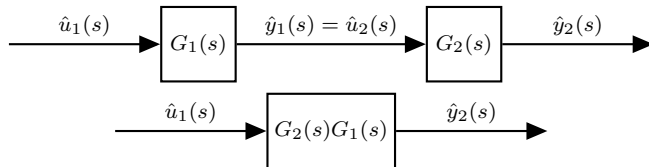
$$\hat{y}_1(s) = G(s)\hat{u}_1(s)$$

$$\hat{y}_2(s) = G(s)\hat{u}_2(s)$$

Cascade interconnection

$$\hat{y}_2(s) = G_2(s)G_1(s)\hat{u}_1(s)$$

Cascade interconnection $\hat{u}_2(s) = \hat{y}_1(s)$



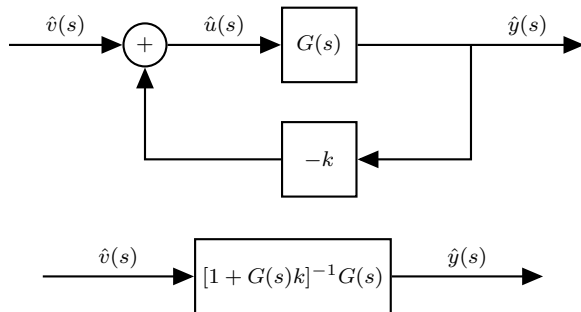
System Interconnections in the Frequency Domain

Consider:

$$\hat{y}(s) = G(s)\hat{u}(s)$$

$$\hat{u}(s) = \hat{v}(s) - k\hat{y}(s)$$

Feedback interconnection:



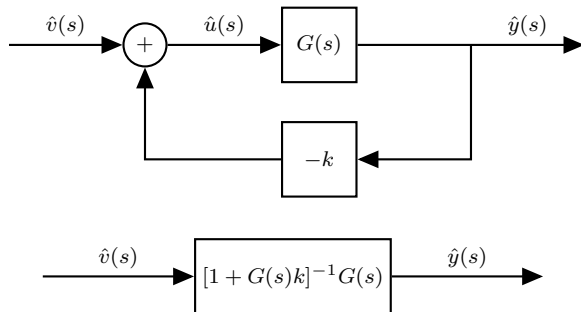
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Feedback interconnection:



Stability and robustness analysis tools:

- Bode diagram
- Nyquist diagram (and Nyquist criterion)

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Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems