Introduction to Nonlinear Control

Stability, control design, and estimation

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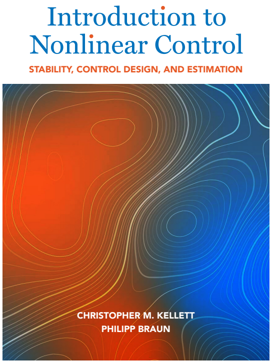

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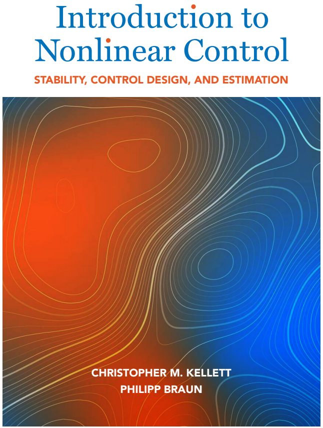

Fundamental Results in the Frequency Domain

Consider single-input single-output (SISO) linear systems:

$$\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t) + du(t),$$

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Notation and assumptions:

- Transfer function $G : \mathbb{C} \to \mathbb{C}$.
- G is a rational function, i.e., there exist polynomial functions $P, Q \in \mathbb{R}[s]$ (with coefficients in $\mathbb{R}$) such that

$$G(s) = \frac{P(s)}{Q(s)}.$$

- P, Q are of minimal degree (i.e., they don't have common zeros)
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- Consider $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ with $\int_0^\infty \psi(t)e^{-st}\,dt < \infty$ for $s \in \mathbb{C}$.

Definition (Laplace transform)

Consider $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. For $s \in \mathcal{C} \subset \mathbb{C}$ for which the integral is well-defined, the Laplace transform $\hat{\psi} : \mathcal{C} \to \mathbb{C}^m$ of ψ is defined as

$$\hat{\psi}(s) \doteq (\mathscr{L}\psi)(s) \doteq \int_0^\infty \psi(t)e^{-st}\,dt.$$

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Definition (Inverse Laplace transform)

Consider $\hat{\varphi} : \mathcal{C} \to \mathbb{C}^m$ and let $\alpha \in \mathbb{R}$ such that $\alpha + j\beta \in \mathcal{C} \subset \mathbb{C}$ for all $\beta \in \mathbb{R}$. Then the inverse Laplace transform $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ of $\hat{\varphi}$ is defined as

$$\varphi(t) = (\mathscr{L}^{-1}\hat{\varphi})(t) = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} e^{st} \hat{\varphi}(s)\, ds$$
$$= \frac{e^{\alpha t}}{2\pi j} \int_{-\infty}^{\infty} e^{jwt} \hat{\varphi}(\alpha + jw)\, dw.$$

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Proposition (Laplace transform properties)

Consider the signals

$$\varphi, \varphi_1, \varphi_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$$

in the time domain and constants

$$a \in \mathbb{R}_{>0}, \ \ a_1, a_2 \in \mathbb{R}.$$

Then the Laplace transform satisfies

$$\begin{aligned}
\mathscr{L}^{-1}\mathscr{L}\varphi(t) &= \varphi(t),\\
\mathscr{L}(a_1\varphi_1 + a_2\varphi_2)(s) &= a_1\hat{\varphi}_1(s) + a_2\hat{\varphi}_2(s),\\
\mathscr{L}(\varphi(a\cdot))(s) &= \tfrac{1}{a}\hat{\varphi}\left(\tfrac{s}{a}\right),\\
\mathscr{L}(\varphi(\cdot - a))(s) &= e^{-sa}\hat{\varphi}(s),\\
\mathscr{L}\left(\tfrac{d^k}{dt^k}\varphi\right)(s) &= s^k\hat{\varphi}(s) - \textstyle\sum_{j=1}^{k-1} s^{j-1}\tfrac{d^{k-1-j}}{dt^{k-1-j}}\varphi(0),\\
\mathscr{L}\left(\int_0^{\cdot} \varphi(\tau)\, d\tau\right)(s) &= \tfrac{1}{s}\hat{\varphi}(s).
\end{aligned}$$

From the time domain to the frequency domain (and back)

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Application of the Laplace transform:

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + b\hat{u}(s), \qquad \hat{y}(s) = c\hat{x}(s) + d\hat{u}(s)$$

Rearrange the terms ($x(0) = 0$):

$$\hat{y}(s) = \left(c(sI - A)^{-1}b + d\right)\hat{u}(s)$$

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Identify input output relationship:

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = c(sI - A)^{-1}b + d \tag{1}$$

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Definition (Realization)

Consider a transfer function $G(s)$ and assume that (1) is satisfied for (A, b, c, d). Then $G(s)$ is called realizable and the quadruple (A, b, c, d) is called a realization of $G(s)$.

Proposition (Laplace transform properties)

Consider the signals $\varphi, \varphi_1, \varphi_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ in the time domain and constants $a \in \mathbb{R}_{>0}$, $a_1, a_2 \in \mathbb{R}$. Then the Laplace transform and its inverse satisfy the following properties:

$$\mathscr{L}^{-1}\mathscr{L}\varphi(t) = \varphi(t),$$

$$\mathscr{L}(a_1\varphi + a_2\varphi_2)(s) = a_1\hat{\varphi}_1(s) + a_2\hat{\varphi}_2(s),$$

$$\mathscr{L}(\varphi(a\cdot))(s) = \tfrac{1}{a}\hat{\varphi}\left(\tfrac{s}{a}\right),$$

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$$\mathscr{L}\left(\int_0^{\cdot}\varphi(\tau)\,d\tau\right)(s) = \tfrac{1}{s}\hat{\varphi}(s).$$

System Interconnections in the Frequency Domain

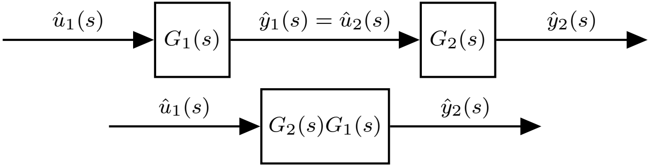
Consider two systems:

$$\hat{y}_1(s) = G(s)\hat{u}_1(s)$$
$$\hat{y}_2(s) = G(s)\hat{u}_2(s)$$

Cascade interconnection

$$\hat{y}_2(s) = G_2(s)G_1(s)\hat{u}_1(s)$$

Cascade interconnection $\hat{u}_2(s) = \hat{y}_1(s)$

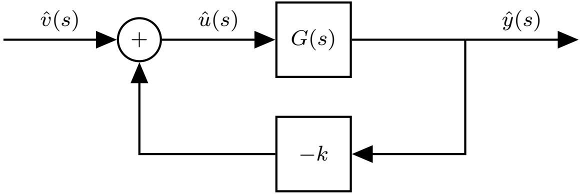


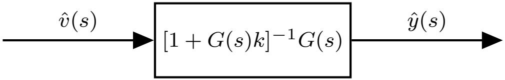
System Interconnections in the Frequency Domain

Consider:

$$\hat{y}(s) = G(s)\hat{u}(s)$$
$$\hat{u}(s) = \hat{v}(s) - k\hat{y}(s)$$

Feedback interconnection:





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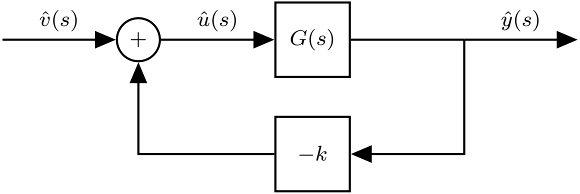

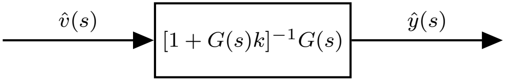

Stability and robustness analysis tools:

- Bode diagram
- Nyquist diagram (and Nyquist criterion)

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