Introduction to Nonlinear Control

Stability, control design, and estimation

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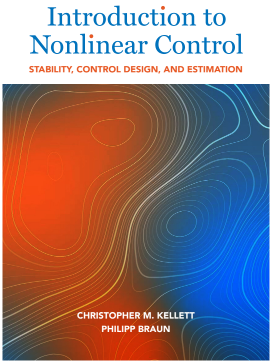
PRINCETON UNIVERSITY PRESS



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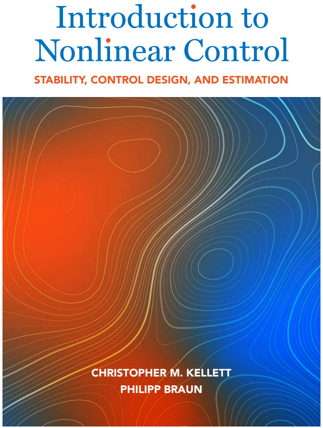

Discrete Time Systems – Fundamentals

Discrete time system (with output):

$$
\begin{aligned}
x_d(k+1) &= F(x_d(k), u_d(k)), \quad x_d(0) = x_{d,0} \in \mathbb{R}^n \\
y_d(k) &= H(x_d(k), u_d(k))
\end{aligned}
$$

Continuous time system (with output):

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n \\
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Time-varying discrete time system ($k \geq k_0 \geq 0$):

$$x_d(k+1) = F(k, x_d(k)), \quad x_d(k_0) = x_{d,0} \in \mathbb{R}^n$$

Time invariant discrete time systems without input:

$$x_d(k+1) = F(x_d(k)), \quad x_d(0) = x_{d,0} \in \mathbb{R}^n,$$

Shorthand notation for difference equations:

$$x_d^+ = F(x_d, u_d),$$

Continuous time system (with output):

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Time-varying continuous time system:

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Definition (Equilibrium)

- The point $x_d^e \in \mathbb{R}^n$ is called equilibrium if $x_d^e = F(x_d^e)$ or $x_d^e = F(k, x_d^e)$ for all $k \in \mathbb{N}$ is satisfied.
- The pair $(x_d^e, u_d^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called equilibrium pair of the system if $x_d^e = F(x_d^e, u_d^e)$ holds.

Continuous time system (with output):

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Definition (Equilibrium)

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Sampling: From Continuous to Discrete Time

Derivative for continuously differentiable function:

$$\tfrac{d}{dt}x(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta) - x(t)}{\Delta}$$

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or equivalently

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u(t))$$

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Approximated discrete time system (identify t with $k \cdot \Delta$)

$$x_d^+ = F(x_d, u_d) \doteq x_d + \Delta f(x_d, u_d)$$

$\rightsquigarrow$ This discretization is known as (explicit) *Euler method*.

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- Continuous time: $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$
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Zero-order hold: for all $k \in \mathbb{N}$, for all $t \in [0, \Delta)$

$$x_d(k) = x(\Delta k) = x(t + \Delta k)$$
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(restrict x and u to piecewise constant functions)

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Sample-and-hold input: (with sampling rate Δ)

$$u(\Delta k) = u(t + \Delta k), \quad k \in \mathbb{N}, \quad \forall\, t \in [0, \Delta)$$

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Sample-and-hold input: (with sampling rate Δ)

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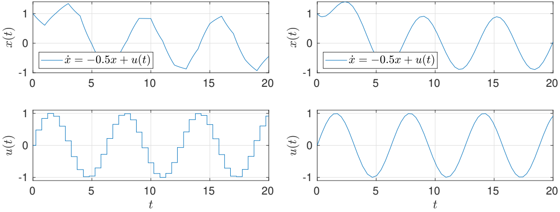
Digital controller:

- apply a piecewise constant sample-and-hold input to a continuous time system.

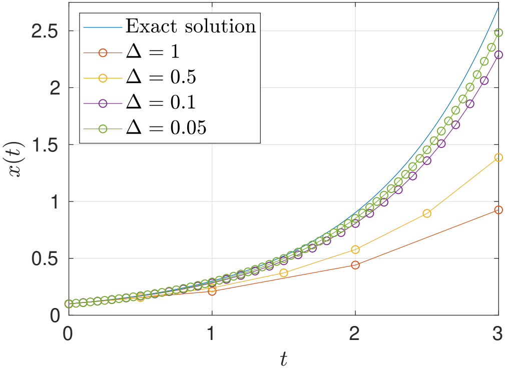
Solution corresponding to sample-and-hold input ($\Delta = 1$) and continuous input



Euler Discretization Example (and Higher Order Discretization Schemes)

Approximation of $\dot{x} = 1.1x$

Euler discretization: $x^+ = (1 + 1.1\Delta)x$



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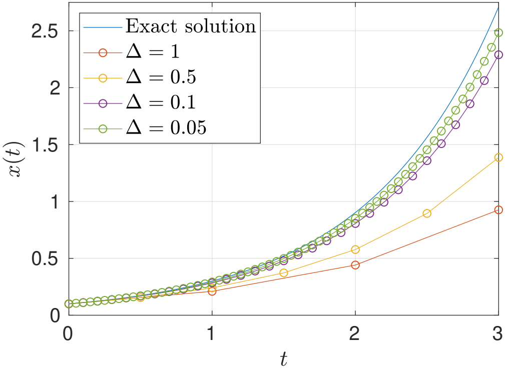
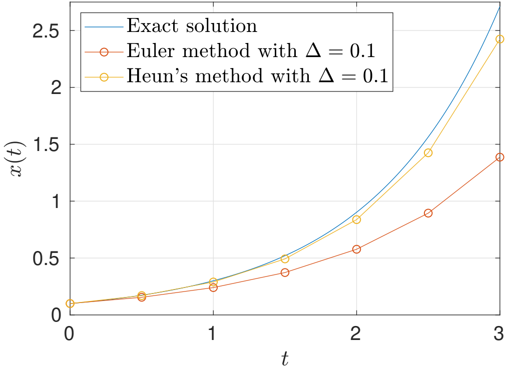


- Euler method:

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u_d)$$

- Heun method:

$$x(t+\Delta) \approx x(t) + \frac{\Delta}{2} f(x(t), u_d) + \frac{\Delta}{2} f(x(t) + \Delta f(x(t), u_d), u_d)$$



Stability Notions

Discrete time systems: Consider

$$x^+ = F(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

Definition

Consider the origin of the discrete time system.

1. (Stability) The origin is *Lyapunov stable* (or simply *stable*) if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x(0)| \leq \delta$ then, for all $k \geq 0$,

$$|x(k)| \leq \varepsilon.$$

2. (Instability) The origin is *unstable* if it is not stable.
3. (Attractivity) The origin is *attractive* if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{k\to\infty} x(k) = 0.$$

4. (Asymptotic stability) The origin is *asymptotically stable* if it is both stable and attractive.

Continuous time systems: Consider

$$\dot{x} = f(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

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Lyapunov Characterizations

Consider $x^+ = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{1}$$
$$V(f(x)) - V(x) \leq 0$$

Then the origin is stable.

Note that

- Decrease condition $V(x^+) = V(f(x)) \leq V(x)$
- differentiability of V (or even continuity) is not required

Consider $\dot{x} = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

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$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2}$$
$$\langle \nabla V(x), f(x) \rangle \leq 0$$

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Theorem (Asymptotic stability)

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{P}$ satisfying $\rho(s) < s$ for all $s > 0$, such that, for all $x \in \mathcal{D}$, (1) holds and

$$V(f(x)) - V(x) \leq -\rho(V(x)).$$

Then the origin is asymptotically stable.

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Linear systems

Consider the discrete time linear system

$$x^+ = Ax, \qquad x(0) \in \mathbb{R}^n \qquad \text{[Solution } x(k) = A^k x(0)]$$

Theorem

The following properties are equivalent:

1. *The origin $x^e = 0$ is exponentially stable;*
2. *The eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of A satisfy $|\lambda_i| < 1$ for all $i = 1, \dots, n$; and*
3. *For $Q \in \mathcal{S}^n_{>0}$ there exists a unique $P \in \mathcal{S}^n_{>0}$ satisfying the discrete time Lyapunov equation*

$$A^T P A - P = -Q.$$

A matrix A which satisfies $|\lambda_i| < 1$ for all $i = 1, \dots, n$ is called a *Schur matrix*.

Consider the continuous time linear system

$$\dot{x} = Ax, \qquad x(0) \in \mathbb{R}^n \qquad \text{[Solution } x(t) = e^{At} x(0)]$$

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$$A^T P + P A = -Q.$$

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$$A^T PA - P = -Q.$$

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If the origin of $z^+ = Az$ with $A = \left[\frac{\partial F}{\partial x}(x)\right]_{x=0}$ is globally exponentially stable, then the origin of $x^+ = F(x)$, $0 = F(0)$, is locally exponentially stable.

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