

Introduction to Nonlinear Control

Stability, control design, and estimation

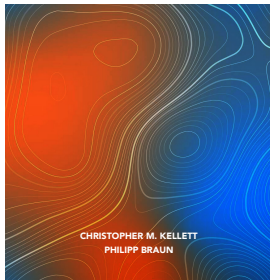
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

CHRISTOPHER M. KELLETT
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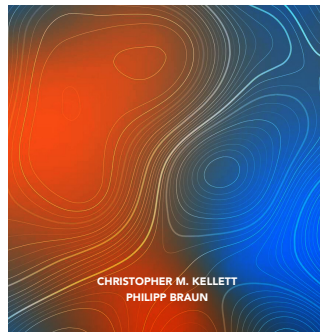
Part I: Dynamical Systems

5 Discrete Time Systems

- 5.1 Discrete Time Systems–Fundamentals
- 5.2 Sampling: From Continuous to Discrete Time
 - 5.2.1 Discretization of Linear Systems
 - 5.2.2 Higher Order Discretization Schemes
- 5.3 Stability Notions
 - 5.3.1 Lyapunov Characterizations
 - 5.3.2 Linear Systems
 - 5.3.3 Stability Preservation of Discretized Systems
- 5.4 Controllability and Observability
- 5.5 Exercises
- 5.6 Bibliographical Notes and Further Reading

Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



Discrete Time Systems – Fundamentals

Discrete time system (with output):

$$\begin{aligned}x_d(k+1) &= F(x_d(k), u_d(k)), & x_d(0) &= x_{d,0} \in \mathbb{R}^n \\ y_d(k) &= H(x_d(k), u_d(k))\end{aligned}$$

Continuous time system (with output):

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), & x(0) &= x_0 \in \mathbb{R}^n \\ y(t) &= h(x(t), u(t))\end{aligned}$$

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Time-varying discrete time system ($k \geq k_0 \geq 0$):

$$x_d(k+1) = F(k, x_d(k)), \quad x_d(k_0) = x_{d,0} \in \mathbb{R}^n$$

Time invariant discrete time systems without input:

$$x_d(k+1) = F(x_d(k)), \quad x_d(0) = x_{d,0} \in \mathbb{R}^n,$$

Shorthand notation for difference equations:

$$x_d^+ = F(x_d, u_d),$$

Continuous time system (with output):

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), & x(0) &= x_0 \in \mathbb{R}^n \\ y(t) &= h(x(t), u(t))\end{aligned}$$

Time-varying continuous time system:

$$\dot{x}(t) = f(t, x(t)), \quad x_d(t_0) = x_0 \in \mathbb{R}^n$$

Time invariant discrete time systems without input:

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n,$$

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Definition (Equilibrium)

- The point $x_d^e \in \mathbb{R}^n$ is called equilibrium if $x_d^e = F(x_d^e)$ or $x_d^e = F(k, x_d^e)$ for all $k \in \mathbb{N}$ is satisfied.
- The pair $(x_d^e, u_d^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called equilibrium pair of the system if $x_d^e = F(x_d^e, u_d^e)$ holds.

Continuous time system (with output):

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Definition (Equilibrium)

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Sampling: From Continuous to Discrete Time

Derivative for continuously differentiable function:

$$\frac{d}{dt}x(t) = \lim_{\Delta \rightarrow 0} \frac{x(t + \Delta) - x(t)}{\Delta}$$

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$$\frac{x(t + \Delta) - x(t)}{\Delta} \approx \frac{d}{dt}x(t) = \dot{x}(t) = f(x(t), u(t))$$

or equivalently

$$x(t + \Delta) \approx x(t) + \Delta f(x(t), u(t))$$

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Approximated discrete time system (identify t with $k \cdot \Delta$)

$$x_d^+ = F(x_d, u_d) \doteq x_d + \Delta f(x_d, u_d)$$

↪ This discretization is known as (explicit) *Euler method*.

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Note that:

- Continuous time: $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ and $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$
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Zero-order hold: for all $k \in \mathbb{N}$, for all $t \in [0, \Delta)$

$$x_d(k) = x(\Delta k) = x(t + \Delta k)$$

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(restrict x and u to piecewise constant functions)

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(restrict x and u to piecewise constant functions)

Sample-and-hold input: (with sampling rate Δ)

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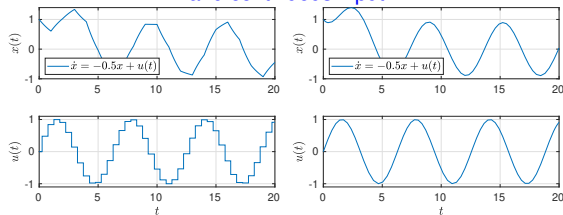
Sample-and-hold input: (with sampling rate Δ)

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Digital controller:

- apply a piecewise constant sample-and-hold input to a continuous time system.

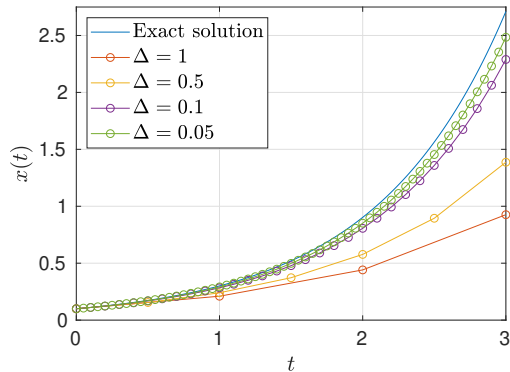
Solution corresponding to sample-and-hold input ($\Delta = 1$) and continuous input



Euler Discretization Example (and Higher Order Discretization Schemes)

Approximation of $\dot{x} = 1.1x$

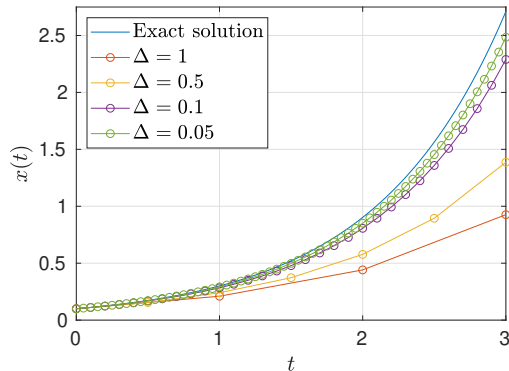
Euler discretization: $x^+ = (1 + 1.1\Delta)x$



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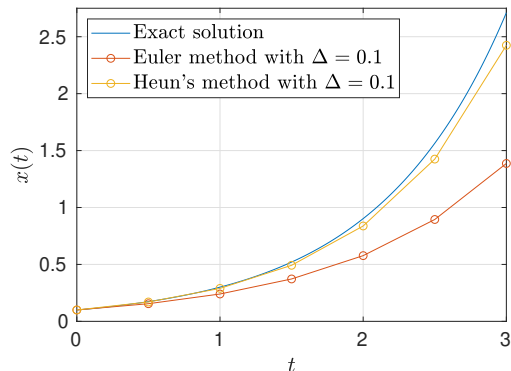


- Euler method:

$$x(t + \Delta) \approx x(t) + \Delta f(x(t), u_d)$$

- Heun method:

$$x(t + \Delta) \approx x(t) + \frac{\Delta}{2} f(x(t), u_d) + \frac{\Delta}{2} f(x(t) + \Delta f(x(t), u_d), u_d)$$



Stability Notions

Discrete time systems: Consider

$$x^+ = F(x), \quad x(0) = x_0 \in \mathbb{R}^n$$

Definition

Consider the origin of the discrete time system.

1. **(Stability)** The origin is *Lyapunov stable* (or simply *stable*) if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x(0)| \leq \delta$ then, for all $k \geq 0$,

$$|x(k)| \leq \varepsilon.$$

2. **(Instability)** The origin is *unstable* if it is not stable.
3. **(Attractivity)** The origin is *attractive* if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{k \rightarrow \infty} x(k) = 0.$$

4. **(Asymptotic stability)** The origin is *asymptotically stable* if it is both stable and attractive.

Continuous time systems: Consider

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n$$

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Lyapunov Characterizations

Consider $x^+ = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

Suppose there exists a continuous function $V : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad (1)$$

$$V(f(x)) - V(x) \leq 0$$

Then the origin is stable.

Note that

- Decrease condition $V(x^+) = V(f(x)) \leq V(x)$
- differentiability of V (or even continuity) is not required

Consider $\dot{x} = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

Suppose there exists a smooth function $V : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad (2)$$

$$\langle \nabla V(x), f(x) \rangle \leq 0$$

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Theorem (Asymptotic stability)

Suppose there exists a continuous function $V : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{P}$ satisfying $\rho(s) < s$ for all $s > 0$, such that, for all $x \in \mathcal{D}$, (1) holds and

$$V(f(x)) - V(x) \leq -\rho(V(x)).$$

Then the origin is asymptotically stable.

Consider $\dot{x} = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

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$$\langle \nabla V(x), f(x) \rangle \leq -\rho(V(x)).$$

Then the origin is asymptotically stable.

Linear systems

Consider the discrete time linear system

$$x^+ = Ax, \quad x(0) \in \mathbb{R}^n \quad [\text{Solution } x(k) = A^k x(0)]$$

Theorem

The following properties are equivalent:

- 1 The origin $x^e = 0$ is *exponentially stable*;
- 2 The eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of A satisfy $|\lambda_i| < 1$ for all $i = 1, \dots, n$; and
- 3 For $Q \in S_{>0}^n$ there exists a unique $P \in S_{>0}^n$ satisfying the *discrete time Lyapunov equation*

$$A^T P A - P = -Q.$$

A matrix A which satisfies $|\lambda_i| < 1$ for all $i = 1, \dots, n$ is called a *Schur matrix*.

Consider the continuous time linear system

$$\dot{x} = Ax, \quad x(0) \in \mathbb{R}^n \quad [\text{Solution } x(t) = e^{At} x(0)]$$

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$$A^T P + P A = -Q.$$

A matrix A which satisfies $\lambda_i \in \mathbb{C}^-$ for all $i = 1, \dots, n$ is called a *Hurwitz matrix*.

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Theorem

If the origin of $z^+ = Az$ with $A = \left[\frac{\partial F}{\partial x}(x) \right]_{x=0}$ is globally exponentially stable, then the origin of $x^+ = F(x)$, $0 = F(0)$, is locally exponentially stable.

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Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems