Introduction to Nonlinear Control

Stability, control design, and estimation

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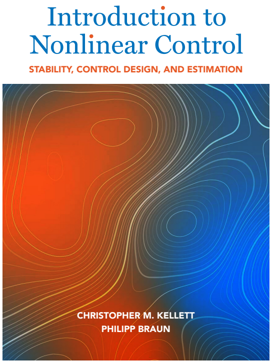
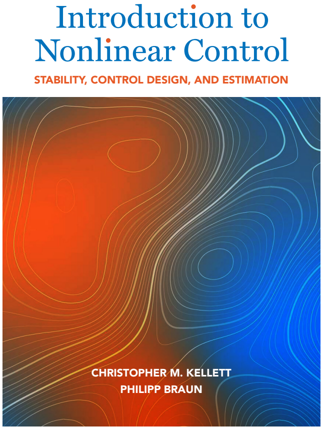
PRINCETON UNIVERSITY PRESS



Part I: Dynamical Systems



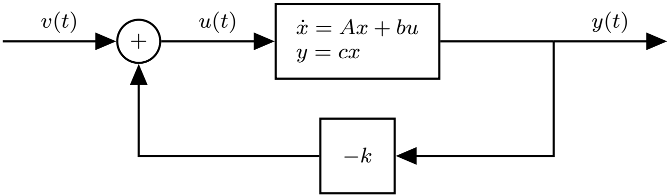
A Commonly Ignored Design Issue

Linear system: $(A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^{n\times 1}, c \in \mathbb{R}^{1\times n})$

$$\dot{x} = Ax + bu, \qquad y = cx,$$

Feedback interconnection: $u = -ky$

$$\dot{x} = (A - bkc)x,$$



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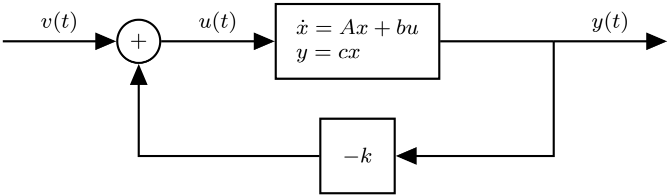

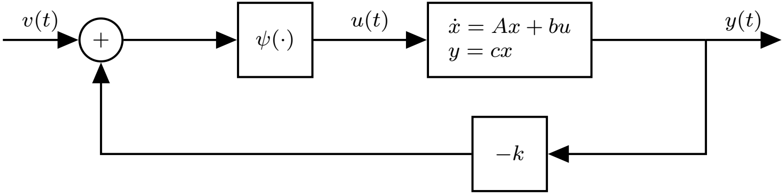

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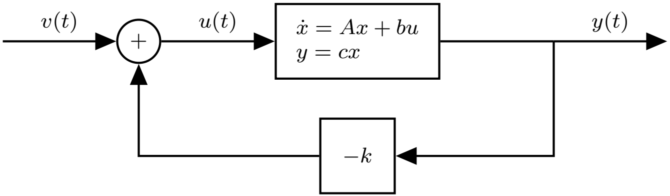
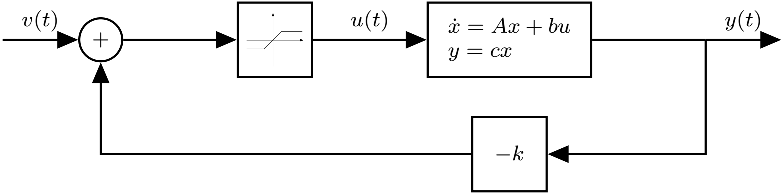


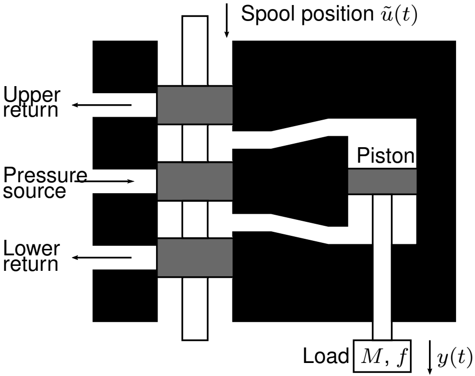


Saturation function

$\mathrm{sat} : \mathbb{R} \to [-1, 1]$:

$$\mathrm{sat}(z) = \begin{cases} -1, & \text{for } z \leq -1 \\ z, & \text{for } z \in [-1, 1] \\ 1, & \text{for } z \geq 1 \end{cases}$$

A Commonly Ignored Design Issue (Example: A servo-valve)



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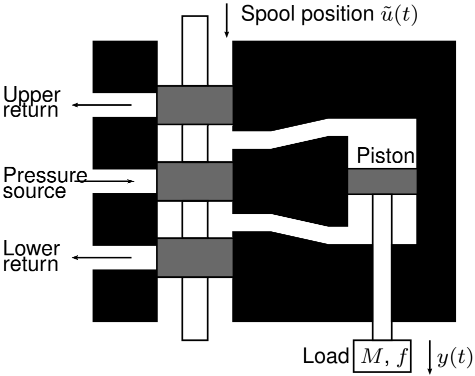
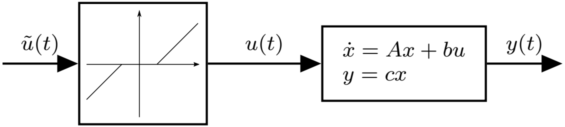




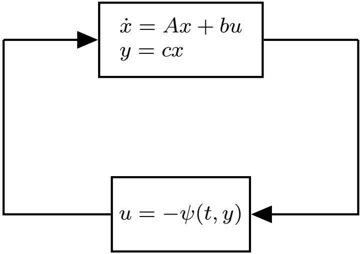
Deadzone function

$\mathrm{dz} : \mathbb{R} \to \mathbb{R}$:

$$\mathrm{dz}(z) = \begin{cases} z + 1, & \text{for } z \leq -1 \\ 0, & \text{for } z \in [-1, 1] \\ y - 1, & \text{for } z \geq 1 \end{cases}$$

A Commonly Ignored Design Issue (The Lur'e Problem)

Consider the feedback interconnection:



Lur'e problem:

- Which conditions on the functions $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ guarantee asymptotic stability of the origin?

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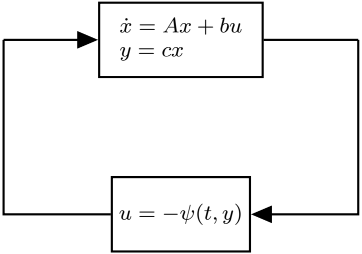


Lur'e problem:

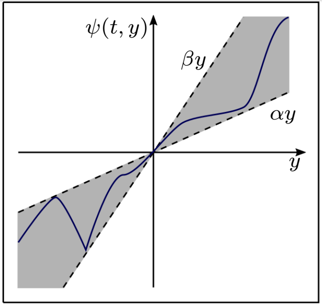
- Which conditions on the functions $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ guarantee asymptotic stability of the origin?

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

for all $t \geq 0$ and for all $y \in \Omega$. For $\Omega = \mathbb{R}$ we say that the sector condition is satisfied globally.



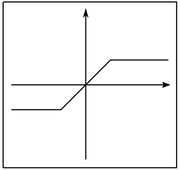
A Commonly Ignored Design Issue (The Sector Condition)

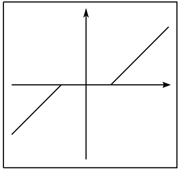
Common nonlinearities: $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$,

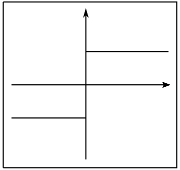
$$\mathrm{sat}(y) = \begin{cases} -1, & \text{for } y \leq -1, \\ y, & \text{for } -1 \leq y \leq 1, \\ 1, & \text{for } y \geq 1. \end{cases}$$

$$\mathrm{dz}(y) = \begin{cases} y+1, & \text{for } y \leq -1, \\ 0, & \text{for } -1 \leq y \leq 1, \\ y-1, & \text{for } y \geq 1. \end{cases}$$

$$\mathrm{sign}(y) = \begin{cases} -1, & \text{for } y < 0, \\ 0, & \text{for } y = 0, \\ 1, & \text{for } y > 0, \end{cases}$$







Question:

- Which nonlinearity satisfies a sector condition?

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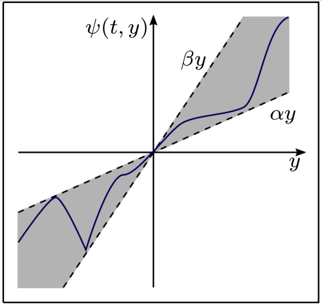

Absolute Stability

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

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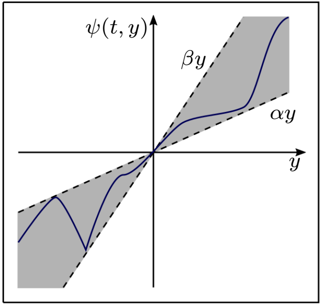

Definition (Absolute stability)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. The Lur'e system

$$\dot{x} = Ax - b\psi(t, y)$$

is called absolutely stable (with respect to α, β, Ω) if the origin is asymptotically stable for all $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfying the sector condition for all $t \geq 0$ and for all $y_0 \in \Omega$.

Historical Perspective on the Lur'e Problem

Conjecture (Aizerman's Conjecture (1949))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, *and suppose the origin of the linear system*

$$\begin{aligned} \dot{x} &= Ax + bu, \\ y &= cx \end{aligned}$$

is globally asymptotically stable for all linear feedbacks

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedbacks in the sector

$$\alpha \leq \frac{\psi(y)}{y} \leq \beta, \quad y \neq 0.$$

$\leadsto$ Conjecture was shown to be wrong through counterexamples.

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Conjecture (Kalman's Conjecture (1957))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system

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$$y = cx$$

is globally asymptotically stable for all linear feedbacks

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedbacks belonging to the incremental sector

$$\alpha \leq \frac{\partial}{\partial y}\psi(y) \leq \beta.$$

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Absolute stability through the Circle Criterion and Popov Criterion

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2 \qquad \forall\, y \in \mathbb{R}, \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Then the system is absolutely stable if:

1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

Definitions: (Disc in the complex plane)

- center $\sigma : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- radius $r : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- for $\alpha \neq 0$ and $\beta > 0$ we define

$$\sigma(\alpha, \beta) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right), \quad r(\alpha, \beta) = \frac{\mathrm{sign}(\alpha)}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$$

Then, the disc $D(\cdot, \cdot)$ is defined as

$$D(\alpha, \beta) = \begin{cases} \{x \in \mathbb{C} : x = -\frac{1}{\beta} + j\omega, \omega \in \mathbb{R}\}, & \alpha = 0 < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha, \beta)| = r(\alpha, \beta)\}, & 0 < \alpha < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha, \beta)| = r(\alpha, \beta)\}, & \alpha < 0 < \beta. \end{cases}$$

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Theorem (Popov Criterion)

Suppose A *is Hurwitz,* (A, b) *is controllable,* (A, c) *is observable, and* $\psi(y)$ *satisfies the sector condition*

$$0 \leq y\psi(y) \leq \beta y^2 \qquad \forall\, y \in \mathbb{R}.$$

Then the Lur'e system with

$$G(s) = c(sI - A)^{-1}b$$

is absolutely stable if there is an $\eta \geq 0$ *with* $-\frac{1}{\eta}$ *not an eigenvalue of* A *such that*

$$H(s) = 1 + (1 + \eta s)\beta G(s)$$

is strictly positive real.

Absolute stability through the Circle Criterion and Popov Criterion

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