Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

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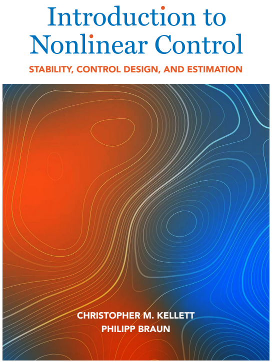

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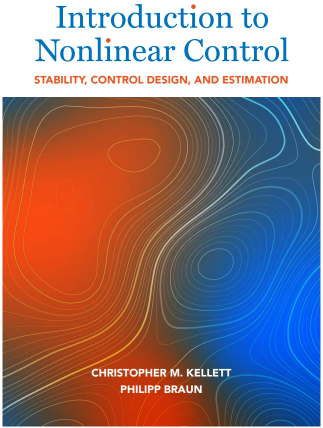

Motivation & Definition

Robust Stability: Consider

$$\dot{x} = Ax + Ew, \quad x(0) = x_0 \in \mathbb{R}^n,$$

with A Hurwitz, and external disturbance w

Recall the solution ($x(t)$, $t \in \mathbb{R}_{\geq 0}$)

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Ew(\tau)d\tau$$

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We can calculate/estimate the impact of the disturbance:

$$\begin{aligned}
|x(t)| &\leq \left|e^{At}x(0)\right| + \left|\int_0^t e^{A(t-\tau)}Ew(\tau)d\tau\right| \\
&\leq \left\|e^{At}\right\| |x(0)| + \int_0^t \left\|e^{A(t-\tau)}\right\| \|E\| |w(\tau)| d\tau \\
&\leq \left\|e^{At}\right\| |x(0)| + \left(\|E\| \int_0^\infty \left\|e^{A\tau}\right\| d\tau\right) \operatorname*{ess\,sup}_{\tau \geq 0} |w(\tau)|
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This bound consists of two components:

- a transient bound; the decaying effect of the initial state $x(0)$
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Input-to-state stability (ISS) for nonlinear systems:

$$\dot{x} = f(x, w), \quad x(0) = x_0 \in \mathbb{R}^n$$

with $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. The set of allowable input functions

$$\mathcal{W} = \{w : \mathbb{R}_{\geq 0} \to \mathbb{R}^m | \ w \text{ essentially bounded}\}.$$

Definition (Input-to-state stability)

The system is said to be *input-to-state stable (ISS)* if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that solutions satisfy

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\|w\|_{\mathcal{L}_\infty}\right)$$

for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \geq 0$.

- $\gamma \in \mathcal{K}$: *ISS-gain*;
- $\beta \in \mathcal{KL}$: *transient bound*.

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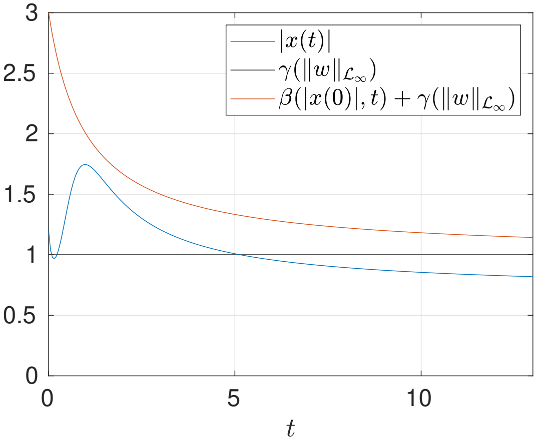

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For linear systems we can conclude that:

- A Hurwitz is sufficient for the system to be ISS.

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Example

Consider the nonlinear/bilinear system:

$$\dot{x} = -x + xw.$$

- The system is 0-input globally asymptotically stable since $w = 0$ implies $\dot{x} = -x$ and so $x(t) = x(0)e^{-t}$
- However, consider the bounded input/disturbance $w = 2$. Then $\dot{x} = x$ and so $x(t) = x(0)e^t$.
- Consequently, it is impossible to find $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|x(t)| = |x(0)|e^t \leq \beta(|x(0)|, t) + \gamma(2).$$

Lyapunov Characterizations

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Theorem (ISS-Lyapunov function)

$\dot{x} = f(x, w)$ is ISS if and only if there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3, \sigma \in \mathcal{K}_\infty$ such that $\forall\, x \in \mathbb{R}^n$, $\forall\, w \in \mathbb{R}^m$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\langle \nabla V(x), f(x, w) \rangle \leq -\alpha_3(|x|) + \sigma(|w|).$$

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Consider

$$\dot{x} = f(x, w) = -x - x^3 + xw, \quad x(0) = x_0 \in \mathbb{R}$$

The candidate ISS-Lyapunov function $V(x) = \frac{1}{2}x^2$ satisfies

$$\begin{aligned} \langle \nabla V(x), f(x, w)\rangle &= \langle x, -x - x^3 + xw\rangle \\ &= -x^2 - x^4 + x^2 w \end{aligned}$$

How to proceed here . . . ?

Lyapunov Characterizations & Young's Inequality

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Detour....

Lemma (Young's inequality)

Let $p, q \in \mathbb{R}_{>0}$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for any* $x, y \in \mathbb{R}^n$ *the inequality*

$$x^T y \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q$$

is satisfied.

Application: Let $p = q = 2$, $\varepsilon > 0$, $a, b \in \mathbb{R}^n$. Then

$$a^T b = (\varepsilon a)^T (\tfrac{1}{\varepsilon} b) \leq \frac{\varepsilon^2}{2}|a|^2 + \frac{1}{2\varepsilon^2}|b|^2$$

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Define $\alpha(s) \doteq s^2 + \frac{1}{2}s^4$ and $\sigma(s) \doteq \frac{1}{2}s^2$, Then

$$\dot{V}(x) \leq -\alpha(|x|) + \sigma(|w|)$$

i.e., V is an ISS-Lyapunov function and the system is ISS.

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i.e., V is an ISS-Lyapunov function and the system is ISS.

$\rightsquigarrow$ Observe that $\dot{x} = -x - x^3 + xw$ is ISS while $\dot{x} = -x + xw$ is not ISS (even though the linearizations are the same)

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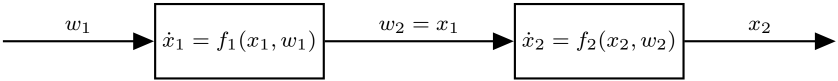
Application: System Interconnection

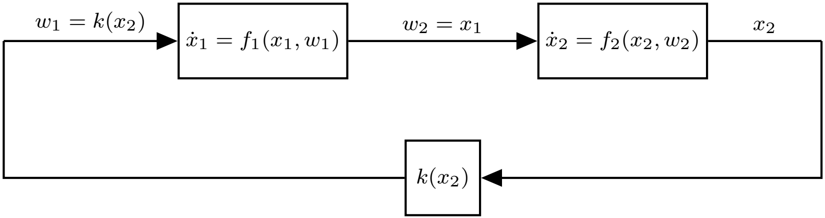
Consider

$$\dot{x}_1 = f_1(x_1, w_1)$$
$$\dot{x}_2 = f_2(x_2, w_2)$$

If system 1 and system 2 are ISS

- is the cascade interconnection ISS?
- is the feedback interconnetion ISS?





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