

Introduction to Nonlinear Control

Stability, control design, and estimation

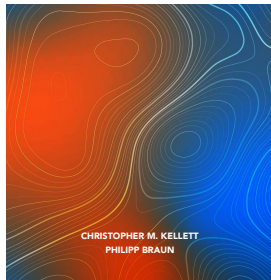
Christopher M. Kellett & Philipp Braun



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STABILITY, CONTROL DESIGN, AND ESTIMATION

CHRISTOPHER M. KELLETT
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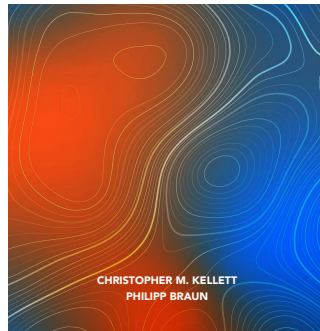
Part I: Dynamical Systems

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Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



Motivation & Definition

Robust Stability: Consider

$$\dot{x} = Ax + Ew, \quad x(0) = x_0 \in \mathbb{R}^n,$$

with A Hurwitz, and **external disturbance** w

Recall the solution $(x(t), t \in \mathbb{R}_{\geq 0})$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Ew(\tau)d\tau$$

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We can calculate/estimate the impact of the disturbance:

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This bound consists of two components:

- a **transient bound**; the decaying effect of the initial state $x(0)$
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Input-to-state stability (ISS) for nonlinear systems:

$$\dot{x} = f(x, w), \quad x(0) = x_0 \in \mathbb{R}^n$$

with $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$. The set of allowable input functions

$$\mathcal{W} = \{w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m \mid w \text{ essentially bounded}\}.$$

Definition (Input-to-state stability)

The system is said to be *input-to-state stable (ISS)* if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that solutions satisfy

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for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \geq 0$.

- $\gamma \in \mathcal{K}$: **ISS-gain**;
- $\beta \in \mathcal{KL}$: **transient bound**.

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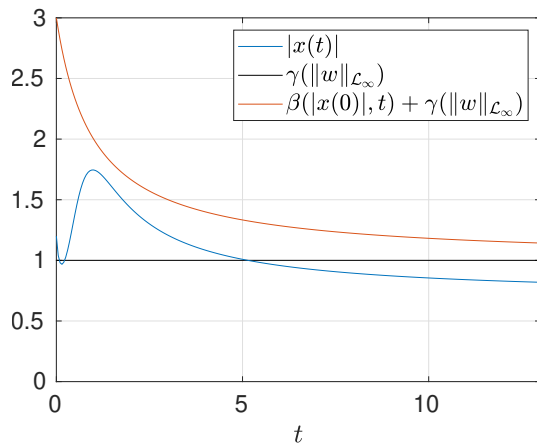
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Example

Consider the nonlinear/bilinear system:

$$\dot{x} = -x + xw.$$

- The system is 0-input globally asymptotically stable since $w = 0$ implies $\dot{x} = -x$ and so $x(t) = x(0)e^{-t}$
- However, consider the bounded input/disturbance $w = 2$. Then $\dot{x} = x$ and so $x(t) = x(0)e^t$.
- Consequently, it is impossible to find $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|x(t)| = |x(0)|e^t \leq \beta(|x(0)|, t) + \gamma(2).$$

Lyapunov Characterizations

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Theorem (ISS-Lyapunov function)

$\dot{x} = f(x, w)$ is ISS if and only if there exist a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and

$\alpha_1, \alpha_2, \alpha_3, \sigma \in \mathcal{K}_\infty$ such that $\forall x \in \mathbb{R}^n, \forall w \in \mathbb{R}^m$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\langle \nabla V(x), f(x, w) \rangle \leq -\alpha_3(|x|) + \sigma(|w|).$$

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Example

Consider

$$\dot{x} = f(x, w) = -x - x^3 + xw, \quad x(0) = x_0 \in \mathbb{R}$$

The candidate ISS-Lyapunov function $V(x) = \frac{1}{2}x^2$ satisfies

$$\begin{aligned} \langle \nabla V(x), f(x, w) \rangle &= \langle x, -x - x^3 + xw \rangle \\ &= -x^2 - x^4 + x^2w \end{aligned}$$

How to proceed here ... ?

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Lyapunov Characterizations & Young's Inequality

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Detour....

Lemma (Young's inequality)

Let $p, q \in \mathbb{R}_{>0}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $x, y \in \mathbb{R}^n$ the inequality

$$x^T y \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q$$

is satisfied.

Application: Let $p = q = 2$, $\varepsilon > 0$, $a, b \in \mathbb{R}^n$. Then

$$a^T b = (\varepsilon a)^T \left(\frac{1}{\varepsilon} b\right) \leq \frac{\varepsilon^2}{2}|a|^2 + \frac{1}{2\varepsilon^2}|b|^2$$

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Define $\alpha(s) \doteq s^2 + \frac{1}{2}s^4$ and $\sigma(s) \doteq \frac{1}{2}s^2$, Then

$$\dot{V}(x) \leq -\alpha(|x|) + \sigma(|w|)$$

i.e., V is an ISS-Lyapunov function and the system is ISS.

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i.e., V is an ISS-Lyapunov function and the system is ISS.

\rightsquigarrow Observe that $\dot{x} = -x - x^3 + xw$ is ISS while $\dot{x} = -x + xw$ is not ISS (even though the linearizations are the same)

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Application: System Interconnection

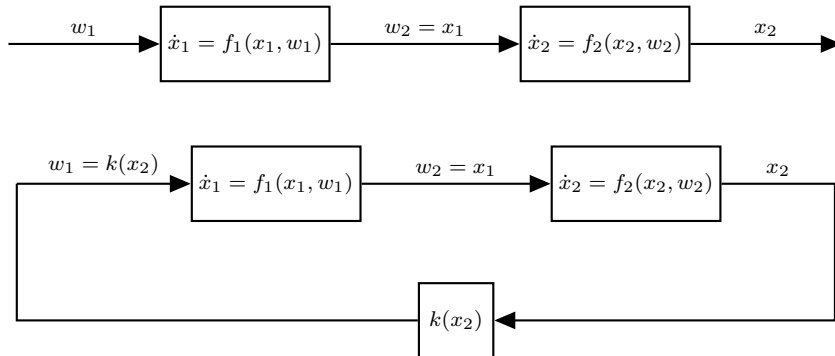
Consider

$$\dot{x}_1 = f_1(x_1, w_1)$$

$$\dot{x}_2 = f_2(x_2, w_2)$$

If system 1 and system 2 are ISS

- is the cascade interconnection ISS?
- is the feedback interconnection ISS?



Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems