Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

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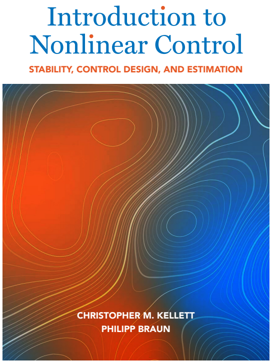
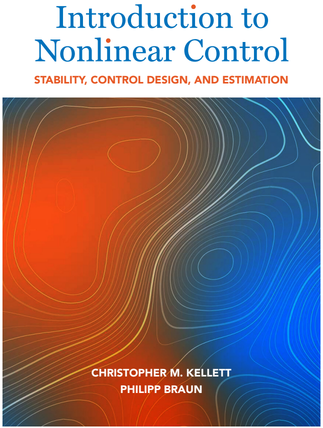


Part II: Controller Design



$\mathcal{L}_2$-Gain Optimization for Linear Systems

Consider: $\dot{x} = Ax$

- Assume that the origin is exponentially stable
- Then, for $Q > 0$ there exists $P > 0$ satisfying

$$A^T P + PA = -Q \quad \text{(Lyapunov equation)}$$

- For $V(x) = x^T Px$ (Lyapunov function) it holds that

$$\dot{V}(x) = x^T(A^T P + PA)x = -x^T Qx < 0, \qquad x \neq 0.$$

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We note that:

- The inequality of the decrease is important not the equality of the Lyapunov equation

⇝ For given A, consider the linear matrix inequality (LMI)

$$0 < P$$
$$A^T P + PA < 0$$

- Advantage: Q is a degree of freedom
- "Optimal" Q and P can be obtained

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LMI (as convex optimization problem):

$$\begin{aligned} \min_{P,\ k} \quad & k \\ \text{subject to} \quad & 0 < k \\ & 0 < P - \alpha I \\ & 0 > P - (k+\alpha)I \\ & 0 > A^T P + PA. \end{aligned}$$

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Here:

- $\alpha > 0$ to ensure that P is not arbitrarily small
- Third constraint to ensure that P is not arbitrarily large

Toolboxes in Matlab:

- CVX, SOSTOOLS, YALMIP

Approximation: $(\varepsilon > 0)$

$$\begin{aligned} \min_{P,\ k} \quad & k \\ \text{subject to} \quad & 0 \leq k \\ & 0 \leq P - \alpha I - \varepsilon I \\ & 0 \geq P - (k+\alpha)I + \varepsilon I \\ & 0 \geq A^T P + PA + \varepsilon I \end{aligned}$$

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization

Consider:

$$\dot{x} = Ax + Ew$$
$$z = Cx + Fw.$$

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Recall: For A Hurwitz, $Q = -2I$, $P > 0$ solution of the Lyap. equation, $V(x) = x^T Px$, we can derive

$$\dot{V}(x) \leq -x^T x + \gamma^2 w^T w, \qquad \gamma = \|PE\|$$

Rearranging terms and integrating (with $x(0) = 0$) yields

$$\begin{aligned} \|x\|^2_{\mathcal{L}_2[0,t)} &\leq \int_0^t x(\tau)^T x(\tau) d\tau + V(x(t)) \\ &\leq \gamma^2 \int_0^t w(\tau)^T w(\tau) d\tau = \gamma^2 \|w\|^2_{\mathcal{L}_2[0,t)}. \end{aligned}$$

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Slight modification: Suppose we can find $P > 0$, so that

$$\dot{V}(x) = x^T(A^T P + PA)x + 2x^T PEw$$
$$< -\gamma\left(\tfrac{1}{\gamma^2} z^T z - w^T w\right), \quad \forall\, (x, w) \neq 0$$

Then we can show that this guarantees

- 0-GAS (since $\dot{V}(x) < 0 \ \ \forall x \neq 0$)
- an $\mathcal{L}_2$-gain bound of $\gamma > 0$ from w to output z; i.e.,

$$\|z\|_{\mathcal{L}_2[0,t)} \leq \gamma \|w\|_{\mathcal{L}_2[0,t)}$$

The bound again follows by integrating (and $x(0) = 0$):

$$\frac{1}{\gamma}\int_0^t z^T(\tau) z(\tau) d\tau + V(x(t)) \leq \gamma \int_0^t w^T(\tau) w(\tau) d\tau$$

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⇝ Can we compute $P > 0$ and $\gamma > 0$ by solving an LMI?

Starting point:

$$x^T(A^T P + PA)x + 2x^T PEw + \tfrac{1}{\gamma} z^T z - \gamma w^T w < 0$$

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization

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$$x^T(A^TP + PA)x + 2x^TPEw + \tfrac{1}{\gamma}z^Tz - \gamma w^Tw < 0$$

$$x^T(A^TP + PA + \tfrac{1}{\gamma}C^TC)x + 2x^T(PE + \tfrac{1}{\gamma}C^TF)w + \gamma w^TF^TFw < 0$$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \left(\begin{bmatrix} A^TP + PA & PE \\ E^TP & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

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In terms of definite matrices ($0 < P$ and):

$$\begin{bmatrix} A^TP + PA & PE \\ E^TP & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} < 0 \qquad (1)$$

Note that:

- For $\gamma > 0$, fixed we know how to solve the LMI to obtain P
- However, we would like to minimize $\gamma > 0$
- The inequality is not linear in γ

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Lemma (Schur Complement)

Let $Q \in \mathcal{S}^r$ and $R \in \mathcal{S}^q$ for $r, q \in \mathbb{N}$ and let $S \in \mathbb{R}^{r \times q}$. Then the matrix condition

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0$$

is equivalent to the matrix conditions

$$R < 0$$

$$Q - SR^{-1}S^T < 0.$$

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Here, take $R = -\gamma$, $S = \begin{bmatrix} C & F \end{bmatrix}$ and Q as the leftmost matrix.

Then, (1) is equivalent to

$$\left[\begin{array}{cc|c} A^TP + PA & PE & C^T \\ E^TP & -\gamma I & F^T \\ \hline C & F & -\gamma I \end{array}\right] < 0$$

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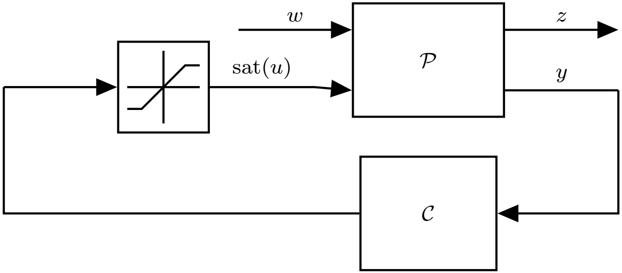
LMI Based Controller and Antiwindup Designs



Plant & Controller:

$$\mathcal{P}: \begin{cases} \dot{x}_p = A_p x_p + B_p \operatorname{sat}(u) + B_w w \\ y = C_{p,y} x_p + D_{p,y} w \\ z = C_{p,z} x_p + D_{p,z} w \end{cases}$$

$$\mathcal{C}: \begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u = C_c x_c + D_{c,y} y \end{cases}$$

Compact representation: ($x = [x_p^T, x_c^T]^T \in \mathbb{R}^n$)

$$\left[\begin{array}{c|c|c} A & B & E \\ \hline C & D & F \\ \hline K & L & G \end{array}\right] = \left[\begin{array}{cc|c|c} A_p + B_p D_{c,y} C_{p,y} & B_p C_c & -B_p & B_p D_{c,y} D_{p,y} + B_w \\ B_c C_{p,y} & A_c & 0 & B_c D_{p,y} \\ \hline C_{p,z} & 0 & 0 & D_{p,z} \\ \hline D_{c,y} C_{p,y} & C_c & 0 & D_{c,y} D_{p,y} \end{array}\right]$$

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \operatorname{sat}(u) \end{aligned}$$

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