

# Introduction to Nonlinear Control

Stability, control design, and estimation

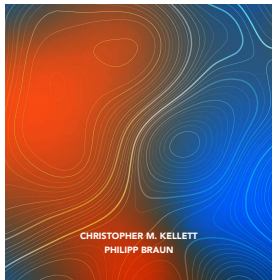
Christopher M. Kellett & Philipp Braun



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STABILITY, CONTROL DESIGN, AND ESTIMATION

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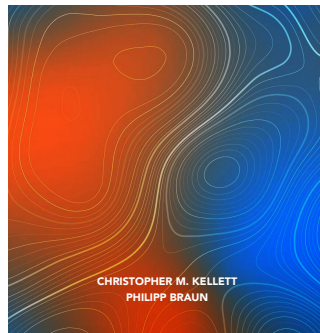
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# Control Lyapunov Functions

Recall the dynamical system consider:

$$\dot{x} = f(x) \quad \text{with} \quad f(0) = 0, \quad x \in \mathbb{R}^n$$

## Theorem (Asymptotic stability theorem)

*Suppose there exists a continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ ,  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$  and  $\rho \in \mathcal{P}$  such that, for all  $x \in \mathbb{R}^n$ ,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

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Control Lyapunov function:  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$

- In terms of a feedback law  $u = k(x)$ ,

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x, k(x)) \rangle < 0, \quad \forall x \neq 0$$

$\leadsto V$  is a Lyapunov function for  $\dot{x} = f(x, k(x)) = \tilde{f}(x)$

- For each  $x \neq 0$  we can find  $u$  such that

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x, u) \rangle < 0$$

## Definition (Control Lyapunov function (CLF))

Let  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ . A continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is called control Lyapunov function for  $\dot{x} = f(x, u)$  if

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n,$$

and for all  $x \in \mathbb{R}^n \setminus \{0\}$  there exists  $u \in \mathbb{R}^m$  such that

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Consider

- a control affine system ( $u \in \mathbb{R}$ )

$$\dot{x} = f(x) + g(x)u$$

with corresponding CLF  $V$ , i.e.,

$$L_f V(x) < 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\} \quad \text{such that} \quad L_g V(x) = 0$$

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Then

- for  $\kappa > 0$  we can define the feedback law

$$k(x) = \begin{cases} - \left( \kappa + \frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^4}}{L_g V(x)^2} \right) L_g V(x), & L_g V(x) \neq 0 \\ 0, & L_g V(x) = 0 \end{cases}$$

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## The feedback law

- asymptotically stabilizes the origin
- inherits the regularity properties of the CLF except at the origin
- is continuous at the origin if the CLF satisfies a small control property (i.e.,  $|k(x)| \rightarrow 0$  for  $|x| \rightarrow 0$ )

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**Note that:** Formula known as

- Universal formula
- Sontag's formula

(Derived by Eduardo Sontag)

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Systems in *strict feedback form*:

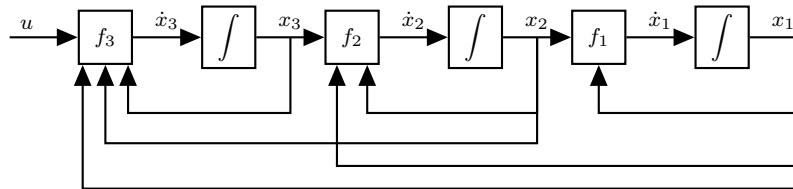
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Example:

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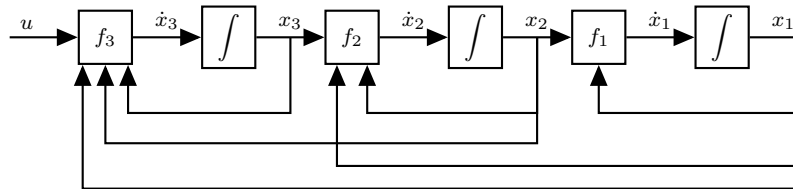
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*Backstepping idea:*

- Treat  $\xi$  as an input to define feedback law  $k_\xi(x)$  stabilizing the  $x$ -dynamics and to find corresponding CLF  $V_1(x)$
- Define error variable  $z = \xi - k_\xi(x)$

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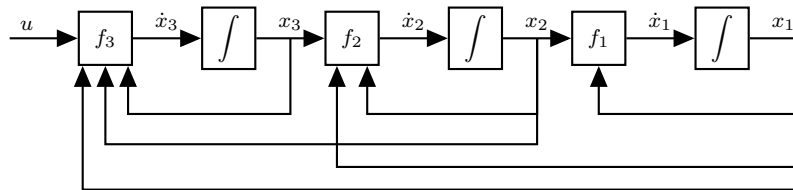
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- Define error variable  $z = \xi - k_\xi(x)$

- Derive error dynamics  $\dot{z} = \frac{d}{dt}(\xi - k_\xi(x))$
- Stabilize error dynamics through feedback law  $k(x, z)$  and define corresponding CLF  $V_2(z)$
- The feedback law stabilizes the original  $(x, \xi)$ -dynamics and a  $V_1(x) + V_2(z)$  is a corresponding CLF.

# Introduction to Nonlinear Control: Stability, control design, and estimation

## Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

## Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

## Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems