

Introduction to Nonlinear Control

Stability, control design, and estimation

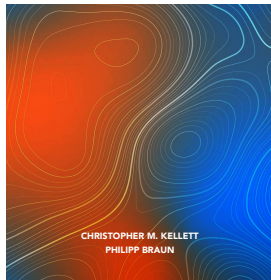
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

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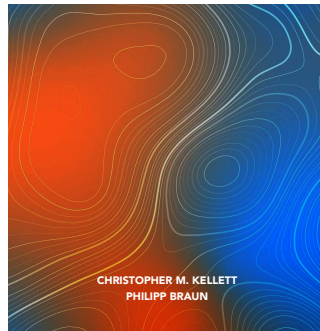
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Setting & Motivation: Sliding Mode Controller Design

We consider systems of the form

$$\dot{x} = f(x, u, \delta(t, x))$$

$$y = h(x)$$

with

- state $x \in \mathbb{R}^n$
- input $u \in \mathbb{R}^m$
- output $y \in \mathbb{R}$
- potentially time and state dependent unknown disturbance $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

We will be interested in

- stabilizing the origin

despite the presence of the disturbance.

↪ First we have to discuss *finite-time stability*.

From Asymptotic Stability ...

Definition (Asymptotic Stability)

Consider $\dot{x} = f(x)$ with $f(0) = 0$.

- The origin is *(Lyapunov) stable* if, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$|x(0)| \leq \delta \quad \text{implies} \quad |x(t)| \leq \varepsilon \quad \forall t \geq 0.$$

- The origin is *attractive* if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

- The origin is *asymptotically stable* for $\dot{x} = f(x)$ if it is both *stable* and *attractive*.

Theorem (Asymptotic stability theorem)

Suppose there exist $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ and $\rho \in \mathcal{P}$ such that, *for all* $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|)$$

Then the origin is (globally) asymptotically stable.

... to Finite-Time Stability

Definition (Finite-time stability)

Consider $\dot{x} = f(x)$ with $f(0) = 0$.

The origin is globally **finite-time stable** if there exists $T : \mathbb{R}^n \setminus \{0\} \rightarrow (0, \infty)$, called the **settling-time function**, such that the following hold:

- **(Stability)**

$\forall \varepsilon > 0 \exists \delta > 0$ such that,

$\forall x(0) = x_0 \in \mathcal{B}_\delta \setminus \{0\}, x(t) \in \mathcal{B}_\varepsilon \forall t \in [0, T(x_0))$.

- **(Finite-time convergence)**

$\forall x(0) = x_0 \in \mathbb{R}^n \setminus \{0\}, x(\cdot)$ is defined on $[0, T(x_0))$, $x(t) \in \mathbb{R}^n \setminus \{0\}$ for all $t \in [0, T(x_0))$, and $x(t) \rightarrow 0$ for $t \rightarrow T(x_0)$.

Theorem (Lyapunov fcn for finite-time stability)

Assume there exist a **continuous function** $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, **which is continuously differentiable on** $\mathbb{R}^n \setminus \{0\}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a constant $\kappa > 0$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle \leq -\kappa \sqrt{V(x)} \quad \forall x \neq 0.$$

Then the origin is globally finite-time stable.

Moreover, the **settling-time** $T(x) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is upper bounded by

$$T(x) \leq \frac{2}{\kappa} \sqrt{\alpha_2(|x|)}.$$

Finite-Time Stability (Example)

Example

Consider

$$\dot{x} = f(x) = -\text{sign}(x) \sqrt[3]{x^2}.$$

We can verify

$$x(t) = \begin{cases} -\frac{1}{27} \text{sign}(x(0))(t - 3\sqrt[3]{|x(0)|})^3 & \text{if } t \leq 3\sqrt[3]{|x(0)|} \\ 0 & \text{if } t \geq 3\sqrt[3]{|x(0)|} \end{cases}$$

Once the equilibrium is reached, the inequalities

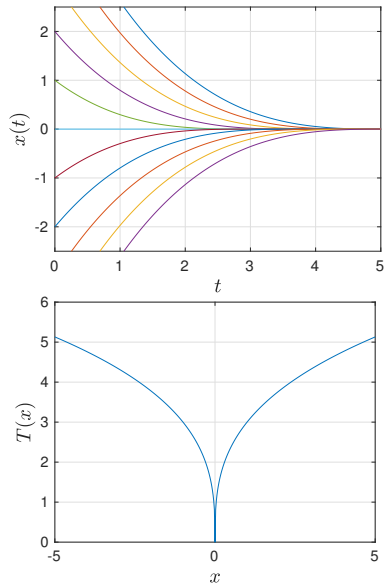
$$-\text{sign}(x) \sqrt[3]{x^2} < 0 \text{ for all } x > 0, \quad \text{and}$$

$$-\text{sign}(x) \sqrt[3]{x^2} > 0 \text{ for all } x < 0$$

ensure that the origin is attractive.

One can show that

- The origin is finite-time stable (with Lyapunov fcn $V(x) = \sqrt[3]{x^2}$)
- Settling time $T(x) = 3\sqrt[3]{|x|}$



Basic Sliding Mode Control

As an example, consider:

$$\begin{aligned}\dot{x} &= x^3 + z, \\ \dot{z} &= u + \delta(t, x, z).\end{aligned}$$

- **Unknown disturbance** $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \rightarrow \mathbb{R}$
- **Assumption:** there exists $L_\delta \in \mathbb{R}_{>0}$ such that

$$|\delta(t, x, z)| \leq L_\delta \quad (t, x, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$$

- Thus, δ is bounded but not necessarily continuous

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Goal: Exponential stability of the x -subsystem

- I.e., we want x to behave as $\dot{x} = -x$ (for all bounded disturbances)
- The desired behavior implies $\dot{x} + x = 0$
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Approach: Define a new state

$$\sigma \doteq x^3 + z + x \quad \text{and} \quad V(\sigma) = \frac{1}{2}\sigma^2$$

- Then

$$\begin{aligned}\dot{V}(\sigma) &= \sigma \dot{\sigma} = \sigma (3x^2 \dot{x} + \dot{z} + \dot{x}) \\ &= \sigma (3x^5 + 3x^2 z + u + \delta(t, x, z) + x^3 + z) .\end{aligned}$$

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- To cancel the known terms define

$$u = v - 3x^5 - 3x^2 z - x^3 - z$$

so that $\dot{V}(\sigma) = \sigma (v + \delta(t, x, z))$ (with new input v)

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- Selecting $v = -\rho \operatorname{sign}(\sigma)$, $\rho > 0$, provides the estimate

$$\begin{aligned}\dot{V}(\sigma) &= \sigma (-\rho \operatorname{sign}(\sigma) + \delta(t, x, z)) = -\rho|\sigma| + \sigma\delta(t, x, z) \\ &\leq -\rho|\sigma| + L_\delta|\sigma| = -(\rho - L_\delta)|\sigma|.\end{aligned}$$

- Finally, with $\rho = L_\delta + \frac{\kappa}{\sqrt{2}}$, $\kappa > 0$, we have

$$\dot{V}(\sigma) \leq -\frac{\kappa|\sigma|}{\sqrt{2}} = -\alpha\sqrt{V(\sigma)} \rightsquigarrow \text{finite-time stab. of } \sigma = 0$$

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- Note that the control

$$u = -\left(L_\delta + \frac{\kappa}{\sqrt{2}}\right) \operatorname{sign}(x^3 + z + x) - 3x^5 - 3x^2 z - x^3 - z$$

is independent of the term $\delta(t, x, z)$.

Basic Sliding Mode Control – Explicit Example

Consider:

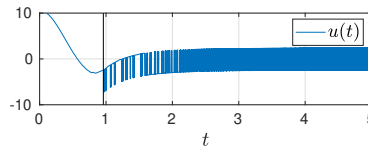
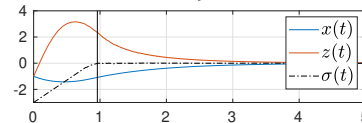
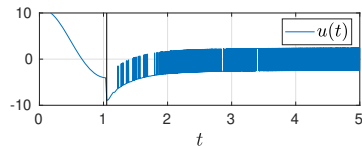
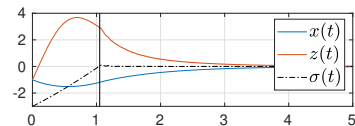
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Control law:

$$u = -\left(L_\delta + \frac{\kappa}{\sqrt{2}}\right) \text{sign}(x^3 + z + x) - 3x^5 - 3x^2z - x^3 - z$$

Parameter selection for the simulations:

- $L_\delta = 1$ and $\kappa = 2$
- $\delta(t, x, z) = \sin(t)$ (top)
- $\delta(t, x, z) = \text{sign}(\cos(2t) \sin(2t))$ (bottom)



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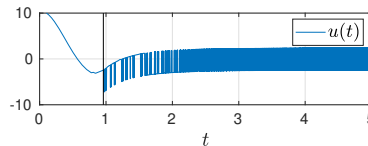
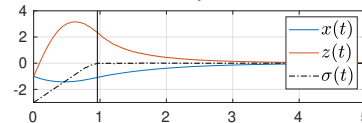
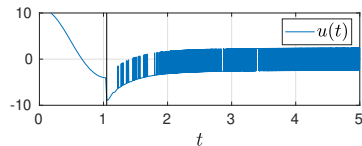
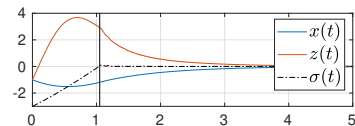
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We observe that

- σ converges to zero in finite-time
- Afterwards (x, z) asymptotically approach the origin
- Since the ordinary differential equation is solved numerically, σ is not exactly zero!



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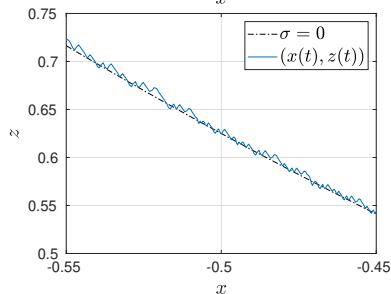
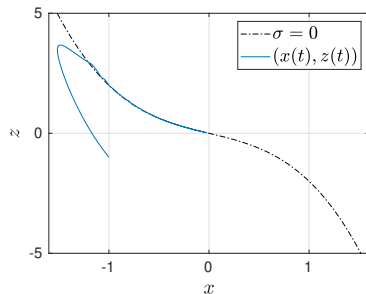
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Convergence structure:

→ Similar to backstepping/forwarding



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Part I: Dynamical Systems

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2. Nonlinear Systems - Stability Notions
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4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

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9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

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17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems