Introduction to Nonlinear Control

Stability, control design, and estimation

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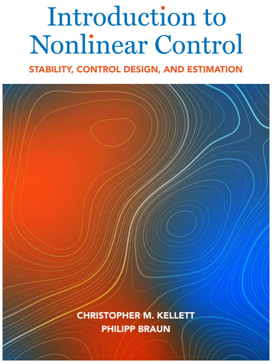

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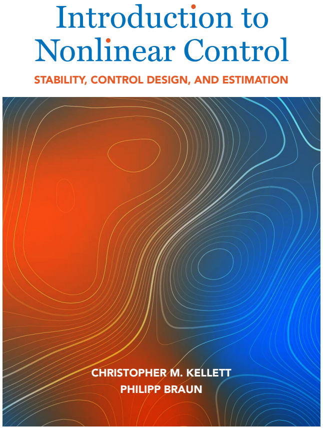
Part II: Controller Design



Motivations and Examples

Consider a parameter-dependent system:

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- Linear controller: For $u = -kx$ it holds that

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- What if a bound on $|\theta|$ is not known?
- Nonlinear controller: $u = -k_1 x - k_2 x^3$, $k_1, k_2 \in \mathbb{R}_{>0}$,

$$\begin{aligned}\dot{x} &= (\theta - k_1)x - k_2 x^3 \\ &= \left[(\theta - k_1) - k_2 x^2\right] x\end{aligned}$$

- It holds that:
  - $\theta \leq k_1$: unique equilibrium in $\mathbb{R}$

$$x^e = 0$$

  - $\theta > k_1$: three equilibria in $\mathbb{R}$

$$x^e \in \left\{0, \pm\sqrt{\tfrac{\theta - k_1}{k_2}}\right\}$$

- Consider $V(x) = \frac{1}{2}x^2$ which satisfies

$$\begin{aligned}\dot{V}(x) &= -k_1 x^2 - k_2 x^4 + \theta x^2 \\ &\leq -k_1 x^2 - (k_2 - \tfrac{1}{2})x^4 + \tfrac{1}{2}\theta^2,\end{aligned}$$

thus it holds that

$$x(t) \overset{t\to\infty}{\to} S_\theta = \left\{ x \in \mathbb{R} \,\middle|\, |x| \leq \sqrt{\tfrac{1}{k_1}}|\theta| \right\}$$

We can conclude that

- Bound on θ known: Global asymptotic stability of 0 can be guaranteed ($k_1 > \theta$)
- Bound on θ not known: Convergence to neighborhood around 0 can be guaranteed

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- Closed loop dynamics

$$\begin{bmatrix} \dot{x} \\ \hline \dot{\xi} \end{bmatrix} = \begin{bmatrix} \theta x - k_1 x - \xi x \\ \hline x^2 \end{bmatrix}$$

- and in terms of error dynamics: $\hat{\theta} = \xi - \theta$

$$\begin{bmatrix} \dot{x} \\ \hline \dot{\hat{\theta}} \end{bmatrix} = \begin{bmatrix} -\hat{\theta} x - k_1 x \\ \hline x^2 \end{bmatrix}$$

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- Consider candidate Lyapunov function

$$V(x, \hat{\theta}) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\hat{\theta}^2$$

which satisfies

$$\begin{aligned} \dot{V}(x, \hat{\theta}) &= (-\hat{\theta}x - k_1 x)x + \hat{\theta}x^2 \\ &= -k_1 x^2 \end{aligned}$$

- This the LaSalle-Yoshizawa theorem implies that
  - $x(t) \overset{t\to\infty}{\to} 0$ for all $(x_0, \xi_0) \in \mathbb{R}^2$
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Dynamic controller designs are can be used to guarantee global convergence properties!

Model Reference Adaptive Control

- Consider linear systems
$$\dot{x} = Ax + Bu$$
with unknown matrices A, B.
- Goal: Design a controller so that the unknown system behaves like
$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u^e$$
    - $\bar{A}$, $\bar{B}$: design parameters
    - u^e: constant reference

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- Control law:

$$u = M(\theta)u^e + L(\theta)x,$$

  - $M(\cdot)$, $L(\cdot)$, to be designed

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- Closed-loop dynamics:
$$\begin{aligned}\dot{x} &= Ax + B(M(\theta)u^e + L(\theta)x)\\ &= (A + BL(\theta))x + BM(\theta)u^e\\ &= A_{\text{cl}}(\theta)x + B_{\text{cl}}(\theta)u^e\end{aligned}$$
where
$$A_{\text{cl}}(\theta) = A + BL(\theta), \qquad B_{\text{cl}}(\theta) = BM(\theta)$$
- Compatibility conditions
$$\begin{aligned} A_{\text{cl}}(\theta) = \bar{A} \quad &\Longleftrightarrow \quad BL(\theta) = \bar{A} - A,\\ B_{\text{cl}}(\theta) = \bar{B} \quad &\Longleftrightarrow \quad BM(\theta) = \bar{B}.\end{aligned}$$
- Overall system dynamics
$$\begin{bmatrix} \dot{x} \\ \hline \dot{\bar{x}} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} (A + BL(\theta))x + BM(\theta)u^e \\ \hline A\bar{x} + Bu^e \\ \Psi(x, \bar{x}, u^e) \end{bmatrix}$$
for Ψ defined appropriately so that $x(t) \to \bar{x}(t)$

Adaptive Backstepping (for Nonlinear Dynamics)

Systems in *parametric strict-feedback form*:

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi_1(x_1)^T\theta\\ \dot{x}_2 &= x_3 + \phi_2(x_1,x_2)^T\theta\\ &\vdots\\ \dot{x}_{n-1} &= x_n + \phi_{n-1}(x_1,\ldots,x_{n-1})^T\theta\\ \dot{x}_n &= \beta(x)u + \phi_n(x)^T\theta\end{aligned}$$

where $\beta(x) \neq 0$ for all $x \in \mathbb{R}^n$

Theorem

Let $c_i > 0$ for $i \in \{1,\ldots,n\}$. Consider the adaptive controller

$$u = \frac{1}{\beta(x)}\alpha_n(x,\vartheta_1,\ldots,\vartheta_n)$$

$$\dot{\vartheta}_i = \Gamma\left(\phi_i(x_1,\ldots,x_i) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j(x_1,\ldots x_j)\right)z_i, \quad i=1,\ldots,n,$$

where $\vartheta_i \in \mathbb{R}^q$ are multiple estimates of θ, $\Gamma > 0$ is the adaptation gain matrix, and the variables z_i and the stabilizing functions

$$\alpha_i = \alpha_i(x_1,\ldots,x_i,\vartheta_1,\ldots,\vartheta_i), \qquad \alpha_i : \mathbb{R}^{i+i\cdot q} \to \mathbb{R}, \qquad i=1,\ldots,n,$$

are defined by the following recursive expressions (and $z_0 \equiv 0$, $\alpha_0 \equiv 0$ for notational convenience)

$$z_i = x_i - \alpha_{i-1}(x_1,\ldots,x_i,\vartheta_1,\ldots,\vartheta_i)$$

$$\alpha_i = -c_i z_i - z_{i-1} - \left(\phi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j\right)^T\vartheta_i$$
$$+ \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \frac{\partial\alpha_{i-1}}{\partial\vartheta_j}\Gamma\left(\phi_j - \sum_{k=1}^{j-1}\frac{\partial\alpha_{j-1}}{\partial x_k}\phi_k\right)z_j\right).$$

This adaptive controller guarantees global boundedness of $x(\cdot)$, $\vartheta_1(\cdot)$, $\ldots$, $\vartheta_n(\cdot)$, and $x_1(t) \to 0$, $x_i(t) \to x_i^e$ for $i=2,\ldots,n$ for $t \to \infty$ where

$$x_i^e = -\theta^T\phi_{i-1}(0,x_2^e,\ldots,x_{i-1}^e), \qquad i=2,\ldots,n.$$

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