

Introduction to Nonlinear Control

Stability, control design, and estimation

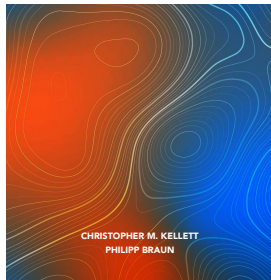
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

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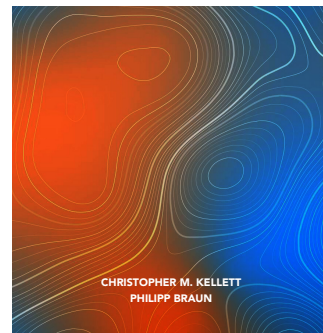
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Motivations and Examples

Consider a parameter-dependent system:

$$\dot{x} = f(x, u, \theta), \quad (\theta \in \mathbb{R}^q \text{ constant but unknown})$$

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- **Linear controller:** For $u = -kx$ it holds that

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- **What if a bound on $|\theta|$ is not known?**
- **Nonlinear controller:** $u = -k_1x - k_2x^3$, $k_1, k_2 \in \mathbb{R}_{>0}$,

$$\begin{aligned} \dot{x} &= (\theta - k_1)x - k_2x^3 \\ &= [(\theta - k_1) - k_2x^2] x \end{aligned}$$

- It holds that:

- ▶ $\theta \leq k_1$: unique equilibrium in \mathbb{R}

$$x^e = 0$$

- ▶ $\theta > k_1$: three equilibria in \mathbb{R}

$$x^e \in \left\{ 0, \pm \sqrt{\frac{\theta - k_1}{k_2}} \right\}$$

- Consider $V(x) = \frac{1}{2}x^2$ which satisfies

$$\begin{aligned} \dot{V}(x) &= -k_1x^2 - k_2x^4 + \theta x^2 \\ &\leq -k_1x^2 - (k_2 - \tfrac{1}{2})x^4 + \tfrac{1}{2}\theta^2, \end{aligned}$$

thus it holds that

$$x(t) \xrightarrow{t \rightarrow \infty} S_\theta = \left\{ x \in \mathbb{R} \mid |x| \leq \sqrt{\tfrac{1}{k_1}|\theta|} \right\}$$

We can conclude that

- **Bound on θ known:** Global asymptotic stability of 0 can be guaranteed ($k_1 > \theta$)
- **Bound on θ not known:** Convergence to neighborhood around 0 can be guaranteed

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- Closed loop dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \theta x - k_1 x - \xi x \\ x^2 \end{bmatrix}$$

- and in terms of error dynamics: $\hat{\theta} = \xi - \theta$

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- Consider candidate Lyapunov function

$$V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}\hat{\theta}^2$$

which satisfies

$$\begin{aligned} \dot{V}(x, \hat{\theta}) &= (-\hat{\theta}x - k_1x)x + \hat{\theta}x^2 \\ &= -k_1x^2 \end{aligned}$$

- This the LaSalle-Yoshizawa theorem implies that

- ▶ $x(t) \xrightarrow{t \rightarrow \infty} 0$ for all $(x_0, \xi_0) \in \mathbb{R}^2$
- ▶ Convergence $\xi(t) \xrightarrow{t \rightarrow \infty} \theta$ is not guaranteed

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Dynamic controller designs can be used to guarantee global convergence properties!

Model Reference Adaptive Control

- Consider linear systems

$$\dot{x} = Ax + Bu$$

with unknown matrices A, B .

- Goal:** Design a controller so that the unknown system behaves like

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u^e$$

- ▶ \bar{A}, \bar{B} : design parameters
- ▶ u^e : constant reference

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- Control law:

$$u = M(\theta)u^e + L(\theta)x,$$

- ▶ $M(\cdot), L(\cdot)$, to be designed

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- Closed-loop dynamics:

$$\begin{aligned}\dot{x} &= Ax + B(M(\theta)u^e + L(\theta)x) \\ &= (A + BL(\theta))x + BM(\theta)u^e \\ &= A_{cl}(\theta)x + B_{cl}(\theta)u^e\end{aligned}$$

where

$$A_{cl}(\theta) = A + BL(\theta), \quad B_{cl}(\theta) = BM(\theta)$$

- Compatibility conditions

$$\begin{aligned}A_{cl}(\theta) &= \bar{A} & \iff & BL(\theta) = \bar{A} - A, \\ B_{cl}(\theta) &= \bar{B} & \iff & BM(\theta) = \bar{B}.\end{aligned}$$

- Overall system dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} (A + BL(\theta))x + BM(\theta)u^e \\ \bar{A}\bar{x} + \bar{B}u^e \\ \Psi(x, \bar{x}, u^e) \end{bmatrix}$$

for Ψ defined appropriately so that $x(t) \rightarrow \bar{x}(t)$

Adaptive Backstepping (for Nonlinear Dynamics)

Systems in *parametric strict-feedback form*:

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi_1(x_1)^T \theta \\ \dot{x}_2 &= x_3 + \phi_2(x_1, x_2)^T \theta \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \phi_{n-1}(x_1, \dots, x_{n-1})^T \theta \\ \dot{x}_n &= \beta(x)u + \phi_n(x)^T \theta\end{aligned}$$

where $\beta(x) \neq 0$ for all $x \in \mathbb{R}^n$

Theorem

Let $c_i > 0$ for $i \in \{1, \dots, n\}$. Consider the adaptive controller

$$u = \frac{1}{\beta(x)} \alpha_n(x, \vartheta_1, \dots, \vartheta_n)$$

$$\dot{\vartheta}_i = \Gamma \left(\phi_i(x_1, \dots, x_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j(x_1, \dots, x_j) \right) z_i, \quad i = 1, \dots, n,$$

where $\vartheta_i \in \mathbb{R}^q$ are multiple estimates of θ , $\Gamma > 0$ is the adaptation gain matrix, and the variables z_i and the stabilizing functions

$$\alpha_i = \alpha_i(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i), \quad \alpha_i : \mathbb{R}^{i+q} \rightarrow \mathbb{R}, \quad i = 1, \dots, n,$$

are defined by the following recursive expressions (and $z_0 \equiv 0$, $\alpha_0 \equiv 0$ for notational convenience)

$$z_i = x_i - \alpha_{i-1}(x_1, \dots, x_i, \vartheta_1, \dots, \vartheta_i)$$

$$\begin{aligned}\alpha_i &= -c_i z_i - z_{i-1} - \left(\phi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j \right)^T \vartheta_i \\ &\quad + \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \vartheta_j} \Gamma \left(\phi_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial x_k} \phi_k \right) z_j \right).\end{aligned}$$

This adaptive controller guarantees global boundedness of $x(\cdot)$, $\vartheta_1(\cdot)$, \dots , $\vartheta_n(\cdot)$, and $x_1(t) \rightarrow 0$, $x_i(t) \rightarrow x_i^e$ for $i = 2, \dots, n$ for $t \rightarrow \infty$ where

$$x_i^e = -\theta^T \phi_{i-1}(0, x_2^e, \dots, x_{i-1}^e), \quad i = 2, \dots, n.$$

Introduction to Nonlinear Control: Stability, control design, and estimation

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4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

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9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

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17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems