Introduction to Nonlinear Control

Stability, control design, and estimation

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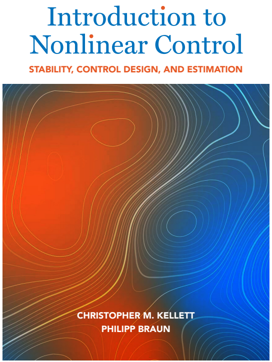
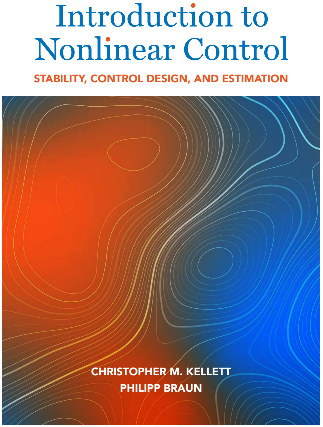


Introduction to Nonlinear Control: Stability, control design, and estimation

Part II: Controller Design



Introductory Examples: (Example 1)

Consider (nonlinear system)

$$\dot{x} = x^3 + u, \quad y = x.$$

Goal:

- Stabilize the state/output $y = x = 0$.

Solution:

- Linear Controller Design:
  - Linearization about the origin:

    $$\dot{x} = u$$

    Natural stabilizing controller selection

    $$u = -kx \qquad (k > 0)$$

    Nonlinear closed loop dynamics:

    $$\dot{x} = x(x^2 - k), \quad x^e \in \{0, \pm\sqrt{k}\}.$$

- Note that:
  - Origin is locally asymptotically stable.
  - Region of attraction increases with k.

- Nonlinear Controller Design:
  - Consider the nonlinear feedback

    $$u = -x^3 + v, \quad v \text{ to be designed}$$

    Closed-loop system

    $$\dot{x} = v$$

    Natural feedback selection

    $$v = -kx \qquad (k > 0)$$

    Closed-loop system with globally asymptotically stable origin k:

    $$\dot{x} = -kx$$

    Overall feedback law:

    $$u = -x^3 - kx$$

- Note that:
  - Coordinate transformation in the input u leads to a linear system.
  - Locally both control laws stabilize the origin, but the coordinate transformation provides a stronger result.

Introductory Examples: (Example 2)

Consider second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 \\ \dot{x}_2 &= -2x_1^3 - 2x_1 x_2 + u \\ y &= x_1, \end{aligned}$$

Consider change of coordinates

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 + x_1^2 \end{aligned}$$

System in new coordinates:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \dot{x}_2 + 2x_1 \dot{x}_1 = u \\ y &= z_1, \end{aligned}$$

Note that: The system is linear in z!

Globally exponentially stabilizing feedback law:

$$\begin{aligned} u &= -k_1 z_1 - k_2 z_2 \\ &= -k_1 x_1 - k_2 \left(x_2 + x_1^2 \right) \end{aligned}$$

for $k_1, k_2 > 0$.

(Can be easily verified by checking the eigenvalues of the closed-loop system.)

Note that:

- Coordinate transformation allows us to stabilize and analyze a linear system instead of a nonlinear system.
- $x \to 0$ is equivalent to $z \to 0$.
- For the input-output behavior it is not important if the dynamics are written in terms of x or z.

Introductory Examples: (Example 3)

Consider third-order nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^3 + u \qquad\qquad y = x_1, \\ \dot{x}_3 &= x_1 + x_3^3\end{aligned}$$

Consider change of coordinates:

$$\begin{aligned}z_1 &= x_3 \\ z_2 &= x_1 + x_3^3 \\ z_3 &= x_2 + 3x_1x_3^2 + 3x_3^5.\end{aligned}$$

and initial feedback (with v to be designed):

$$u = -x_3^3 - 3x_2x_3^2 - 6x_1^2x_3 - 21x_1x_3^4 - 15x_3^7 + v,$$

Leads to linear states (but a nonlinear output):

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \qquad\qquad y = z_2 - z_1^3. \\ \dot{z}_3 &= v\end{aligned}$$

The feedback law

$$u = -x_3^3 + v$$

Leads to linear input-output relationship from v to y:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= v \qquad\qquad y = x_1, \\ \dot{x}_3 &= x_1 + x_3^3\end{aligned}$$

Here,

- the "internal" x_3 dynamics, which are not visible through the output, are nonlinear.
- we are able to partially linearize the dynamics
- For the linear dynamics, v can be defined such that the origin $z = 0$ is asymptotically stable (i.e., y converges to zero).
- For the nonlinear dynamics a controller guaranteeing $y(t) \to 0$ for $t \to \infty$ can be defined using pole placement. But is the origin asymptotically stable?

In this chapter we will discuss . . .

Feedback linearization

- Input-to-state linearization
- Input-to-output linearization

Relies on properties such as

- relative degree
- zero dynamics

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Relies on concepts such as

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- coordinate transformation of the input
- (repeated) Lie derivatives ($\lambda : \mathbb{R}^n \to \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$)

$$L_f \lambda(x) = \frac{\partial \lambda}{\partial x}(x) \cdot f(x)$$

$$L_f^k \lambda(x) = \frac{\partial}{\partial x}\left(L_f^{k-1}\lambda(x)\right) f(x), \quad L_f^0 \lambda(x) = \lambda(x)$$

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This also allows us to talk about

- nonlinear controllability (accessibility)

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