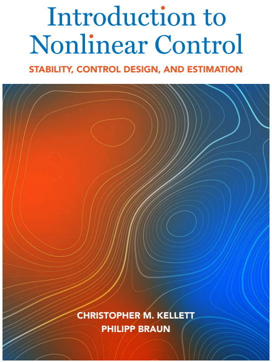
Introduction to Nonlinear Control

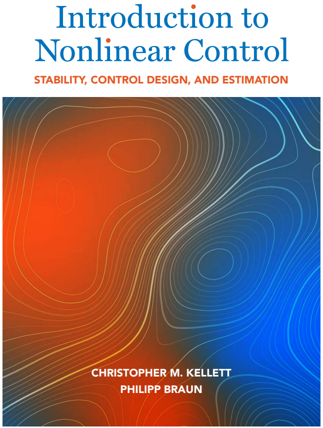
Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

PRINCETON UNIVERSITY PRESS



Part II: Controller Design



Optimal Control – Continuous Time Setting

Consider

- continuous time system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n \qquad (1)$$

  with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$.
- *cost functional* (or performance criterion)

$$J(x_0, u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))d\tau \in \mathbb{R} \cup \{\pm\infty\}$$

  defined through *running costs* $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

This defines an

- *(optimal) value function*: $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$

$$\text{subject to (1).}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

We hope to find a feedback law: ($\mu : \mathbb{R}^n \to \mathbb{R}^m$)

$$\mu(x^\star(t)) = u^\star(t) \qquad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Here

- $(x^\star(\cdot), u^\star(\cdot))$ is an optimal solution pair
- $x^\star(\cdot)$ uniquely defined through $u^\star(\cdot)$ and $x^\star(0) = x_0$

Optimal Control – Continuous Time Setting

Consider

- continuous time system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n \qquad (1)$$

  with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$.

- *cost functional* (or performance criterion)

$$J(x_0, u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau)) d\tau \in \mathbb{R} \cup \{\pm\infty\}$$

  defined through *running costs* $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

This defines an

- *(optimal) value function*: $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$

$$\text{subject to (1).}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

We hope to find a feedback law: ($\mu : \mathbb{R}^n \to \mathbb{R}^m$)

$$\mu(x^\star(t)) = u^\star(t) \qquad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Here

- $(x^\star(\cdot), u^\star(\cdot))$ is an optimal solution pair
- $x^\star(\cdot)$ uniquely defined through $u^\star(\cdot)$ and $x^\star(0) = x_0$

Example:

- linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

- with quadratic cost function: ($Q \geq 0$, $R > 0$)

$$J(x_0, u(\cdot)) = \int_0^\infty \left(x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \right) d\tau$$

Unfortunately:

- for general nonlinear systems and running costs, the optimization problem is usually intractable.

Optimal Control – Linear Quadratic Regulator

Consider

- continuous time system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n \qquad (1)$$

  with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$.
- *cost functional* (or performance criterion)

$$J(x_0, u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau)) d\tau \in \mathbb{R} \cup \{\pm\infty\}$$

  defined through *running costs* $\ell(x, u) \geq 0$

This defines an

- *(optimal) value function*: $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$

$$\text{subject to (1).}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

The Linear Quadratic Regulator (Example)

Linear system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

Quadratic cost function:

$$J(x_0, u(\cdot)) = \int_0^\infty \left(x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \right) d\tau$$

Theorem (Linear quadratic regulator (LQR))

Let $Q \geq 0$, $R > 0$. If there exists $P > 0$ satisfying the continuous time algebraic Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

and if $$A - BR^{-1}B^T P$$

is Hurwitz, then the optimal feedback law and value function are given by

$$\mu(x) = -R^{-1}B^T P x$$

$$V(x_0) = x_0^T P x_0.$$

Optimal Control – Discrete Time Setting

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (2)$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)) \in \{\pm\infty\}$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- (optimal) value function:

$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}} J(x_0, u(\cdot))$$
$$\text{subject to (2)}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

- Optimal solution pair: $(x^\star(\cdot), u^\star(\cdot))$

Optimal Control – Discrete Time Setting

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (2)$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)) \in \{\pm\infty\}$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- (optimal) value function:

$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}} J(x_0, u(\cdot))$$
$$\text{subject to (2)}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

- Optimal solution pair: $(x^\star(\cdot), u^\star(\cdot))$

The Linear Quadratic Regulator (Example)

Linear system:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \in \mathbb{R}^n$$

Quadratic cost function:

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \Big(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \Big) \, d\tau$$

Theorem (Linear quadratic regulator (LQR))

Let $Q \geq 0$, $R > 0$. If there exists $P > 0$ satisfying the discrete time algebraic Riccati equation

$$Q + A^T PA - P - A^T PB(R + B^T PB)^{-1} B^T PA = 0$$

and if $\quad A - B(R + B^T PB)^{-1} B^T PA$

is Schur, then the optimal feedback law and value function are given by

$$\mu(x) = -(R + B^T PB)^{-1} B^T PAx$$
$$V(x_0) = x_0^T P x_0.$$

From Infinite to Finite Dimensional Optimization

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (2)$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)) \in \{\pm\infty\}$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- (optimal) value function:

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$
$$\text{subject to (2)}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

- Optimal solution pair: $(x^\star(\cdot), u^\star(\cdot))$

From Infinite to Finite Dimensional Optimization

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (2)$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)) \in \{\pm\infty\}$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- (optimal) value function:

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$

subject to (2)

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

- Optimal solution pair: $(x^\star(\cdot), u^\star(\cdot))$

Recapitulation of results:

- $(x^\star(\cdot), u^\star(\cdot))$ is optimal with respect to a specific measure (i.e., a specific cost functional).
- To obtain the optimal solution pair an infinite dimensional optimization problem needs to be solved.

From Infinite to Finite Dimensional Optimization

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \tag{2}$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)) \in \{\pm\infty\}$$

  (with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- (optimal) value function:

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$
$$\text{subject to (2)}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

- Optimal solution pair: $(x^\star(\cdot), u^\star(\cdot))$

Recapitulation of results:

- $(x^\star(\cdot), u^\star(\cdot))$ is optimal with respect to a specific measure (i.e., a specific cost functional).
- To obtain the optimal solution pair an infinite dimensional optimization problem needs to be solved.

⇝ In general, the value function and the solution pair can only be calculated under specific assumptions (e.g. linear dynamics)

From Infinite to Finite Dimensional Optimization

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (2)$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)) \in \{\pm\infty\}$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- (optimal) value function:

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$

subject to (2)

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

- Optimal solution pair: $(x^\star(\cdot), u^\star(\cdot))$

Recapitulation of results:

- $(x^\star(\cdot), u^\star(\cdot))$ is optimal with respect to a specific measure (i.e., a specific cost functional).
- To obtain the optimal solution pair an infinite dimensional optimization problem needs to be solved.

⇝ In general, the value function and the solution pair can only be calculated under specific assumptions (e.g. linear dynamics)

⇝ In general, only open loop solutions $(x^\star(\cdot), u^\star(\cdot))$ (instead of optimal feedback law $\mu(x(t))$ are obtained)

From Infinite to Finite Dimensional Optimization

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (2)$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)) \in \{\pm\infty\}$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- (optimal) value function:

$$V(x_0) = \min_{u(\cdot)} J(x_0, u(\cdot))$$

$$\text{subject to (2)}$$

- optimal input trajectory:

$$u^\star(\cdot) = \arg\min_{u(\cdot)} J(x_0, u(\cdot)).$$

- Optimal solution pair: $(x^\star(\cdot), u^\star(\cdot))$

Recapitulation of results:

- $(x^\star(\cdot), u^\star(\cdot))$ is optimal with respect to a specific measure (i.e., a specific cost functional).
- To obtain the optimal solution pair an infinite dimensional optimization problem needs to be solved.

⇝ In general, the value function and the solution pair can only be calculated under specific assumptions (e.g. linear dynamics)

⇝ In general, only open loop solutions $(x^\star(\cdot), u^\star(\cdot))$ (instead of optimal feedback law $\mu(x(t))$ are obtained)

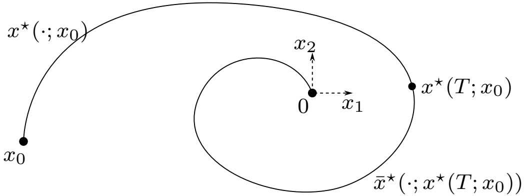
How can we

- obtain results for nonlinear systems?
- incorporate state/input constraints?
- simplify the infinite horizon (or infinite dimensional) optimization problem?

From Infinite to Finite Dimensional Optimization: The Principle of Optimality

The principle of optimality:

- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.



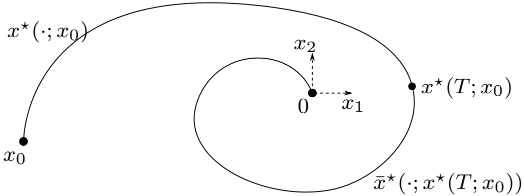
From Infinite to Finite Dimensional Optimization: The Principle of Optimality

The principle of optimality:

- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.

More formally:

- Assume that solutions of the optimal control problem are unique.



From Infinite to Finite Dimensional Optimization: The Principle of Optimality

The principle of optimality:

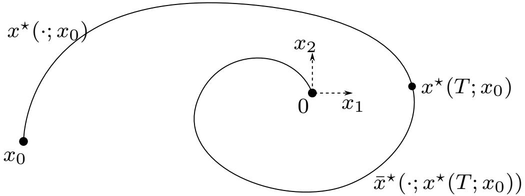
- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.

More formally:

- Assume that solutions of the optimal control problem are unique.
- For $x_0 \in \mathbb{R}^n$ let $(x^\star(\cdot; x_0), u^\star(\cdot; x_0))$ be the optimal solution pair of

$$u^\star(\cdot; x_0) = \arg\min_{u(\cdot)} J(x_0, u(\cdot))$$

$$\text{subject to dynamics \& initial cond.}$$



From Infinite to Finite Dimensional Optimization: The Principle of Optimality

The principle of optimality:

- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.

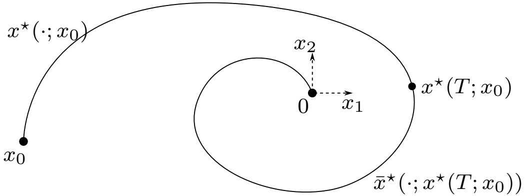
More formally:

- Assume that solutions of the optimal control problem are unique.
- For $x_0 \in \mathbb{R}^n$ let $(x^\star(\cdot; x_0), u^\star(\cdot; x_0))$ be the optimal solution pair of

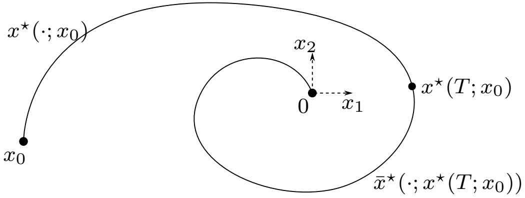
$$u^\star(\cdot; x_0) = \arg\min_{u(\cdot)} J(x_0, u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$

- For any $T \geq 0$ let $(\bar{x}^\star(\cdot; x^\star(T; x_0)), \bar{u}^\star(\cdot; x^\star(T; x_0)))$ be the optimal solution pair of

$$\bar{u}^\star(\cdot; x^\star(T; x_0)) = \arg\min_{u(\cdot)} J(x^\star(T; x_0), u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$



From Infinite to Finite Dimensional Optimization: The Principle of Optimality



The principle of optimality:

- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.

More formally:

- Assume that solutions of the optimal control problem are unique.
- For $x_0 \in \mathbb{R}^n$ let $(x^\star(\cdot; x_0), u^\star(\cdot; x_0))$ be the optimal solution pair of

$$u^\star(\cdot; x_0) = \arg\min_{u(\cdot)} J(x_0, u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$

- For any $T \geq 0$ let $(\bar{x}^\star(\cdot; x^\star(T; x_0)), \bar{u}^\star(\cdot; x^\star(T; x_0)))$ be the optimal solution pair of

$$\bar{u}^\star(\cdot; x^\star(T; x_0)) = \arg\min_{u(\cdot)} J(x^\star(T; x_0), u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$

- Then the principle of optimality states that

$$\bar{u}^\star(\cdot; x^\star(T; x_0)) = u^\star(\cdot + T; x_0)$$
$$\bar{x}^\star(\cdot; x^\star(T; x_0)) = x^\star(\cdot + T; x_0)$$

From Infinite to Finite Dimensional Optimization: The Principle of Optimality

The principle of optimality:

- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.

More formally:

- Assume that solutions of the optimal control problem are unique.
- For $x_0 \in \mathbb{R}^n$ let $(x^\star(\cdot; x_0), u^\star(\cdot; x_0))$ be the optimal solution pair of

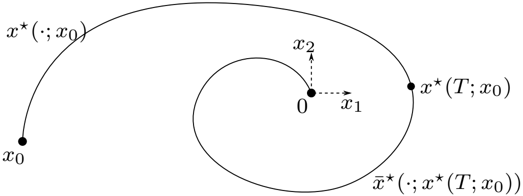
$$u^\star(\cdot; x_0) = \arg\min_{u(\cdot)} J(x_0, u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$

- For any $T \geq 0$ let $(\bar{x}^\star(\cdot; x^\star(T; x_0)), \bar{u}^\star(\cdot; x^\star(T; x_0)))$ be the optimal solution pair of

$$\bar{u}^\star(\cdot; x^\star(T; x_0)) = \arg\min_{u(\cdot)} J(x^\star(T; x_0), u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$

- Then the principle of optimality states that

$$\bar{u}^\star(\cdot; x^\star(T; x_0)) = u^\star(\cdot + T; x_0)$$
$$\bar{x}^\star(\cdot; x^\star(T; x_0)) = x^\star(\cdot + T; x_0)$$



Note that:

- In the case that optimal solutions are not unique, 'only'

$$J(x^\star(T; x), \bar{u}^\star(\cdot; x^\star(T; x))) = J(x^\star(T; x), u^\star(\cdot + T; x))$$

is guaranteed.

From Infinite to Finite Dimensional Optimization: The Principle of Optimality

The principle of optimality:

- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.

More formally:

- Assume that solutions of the optimal control problem are unique.
- For $x_0 \in \mathbb{R}^n$ let $(x^\star(\cdot;x_0), u^\star(\cdot;x_0))$ be the optimal solution pair of

$$u^\star(\cdot;x_0) = \arg\min_{u(\cdot)} J(x_0, u(\cdot))$$

$$\text{subject to dynamics \& initial cond.}$$

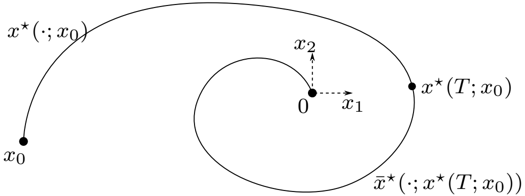
- For any $T \geq 0$ let $(\bar{x}^\star(\cdot;x^\star(T;x_0)), \bar{u}^\star(\cdot;x^\star(T;x_0)))$ be the optimal solution pair of

$$\bar{u}^\star(\cdot;x^\star(T;x_0)) = \arg\min_{u(\cdot)} J(x^\star(T;x_0), u(\cdot))$$

$$\text{subject to dynamics \& initial cond.}$$

- Then the principle of optimality states that

$$\bar{u}^\star(\cdot;x^\star(T;x_0)) = u^\star(\cdot + T;x_0)$$
$$\bar{x}^\star(\cdot;x^\star(T;x_0)) = x^\star(\cdot + T;x_0)$$

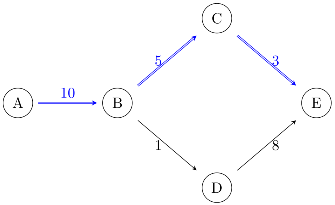


Note that:

- In the case that optimal solutions are not unique, 'only'

$$J(x^\star(T;x), \bar{u}^\star(\cdot;x^\star(T;x))) = J(x^\star(T;x), u^\star(\cdot + T;x))$$

  is guaranteed.
- Same result in the discrete time setting



Constrained Optimal Control for Linear Systems

- Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k)$$

Now:

- We restrict the input space to

$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m |\; u(k) \in \mathbb{U}\; \forall k \in \mathbb{N}\},$$

  for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \operatorname{int} \mathbb{U}$

- Corresponding OCP:

$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$

$$\text{subject to dynamics \& initial cond.}$$

Constrained Optimal Control for Linear Systems

- Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k)$$

Now:

- We restrict the input space to

$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m \,|\, u(k) \in \mathbb{U} \; \forall k \in \mathbb{N}\},$$

  for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \operatorname{int} \mathbb{U}$

- Corresponding OCP:

$$V(x_0) = \min_{u(\cdot) \in \mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$

$$\text{subject to dynamics \& initial cond.}$$

- Step 1: Apply the results of the unconstrained setting (i.e., $\mathbb{U} = \mathbb{R}^m$) to obtain Lyapunov function $V(x) = x^T P_F x$ and the optimal feedback law

$$\mu(x) = Kx = -\left(R + B^T P B\right)^{-1} B^T P A x.$$

Constrained Optimal Control for Linear Systems

- Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k)$$

Now:

- We restrict the input space to

$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m |\ u(k) \in \mathbb{U}\ \forall k \in \mathbb{N}\},$$

for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \operatorname{int} \mathbb{U}$

- Corresponding OCP:

$$V(x_0) = \min_{u(\cdot) \in \mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$

$$\text{subject to dynamics \& initial cond.}$$

- Step 1: Apply the results of the unconstrained setting (i.e., $\mathbb{U} = \mathbb{R}^m$) to obtain Lyapunov function $V(x) = x^T P_F x$ and the optimal feedback law

$$\mu(x) = Kx = -\left(R + B^T PB\right)^{-1} B^T PAx.$$

- Since V is a Lyapunov function for $x^+ = (A + BK)x$ the sublevel set $\mathbb{X}_F = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is forward invariant for $c > 0$

Constrained Optimal Control for Linear Systems

- Consider
$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$
- Cost functional
$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k)$$

Now:

- We restrict the input space to
$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m |\; u(k) \in \mathbb{U} \; \forall k \in \mathbb{N}\},$$
for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \operatorname{int} \mathbb{U}$
- Corresponding OCP:
$$V(x_0) = \min_{u(\cdot) \in \mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$
- Step 1: Apply the results of the unconstrained setting (i.e., $\mathbb{U} = \mathbb{R}^m$) to obtain Lyapunov function $V(x) = x^T P_F x$ and the optimal feedback law
$$\mu(x) = Kx = -\left(R + B^T PB\right)^{-1} B^T PAx.$$

- Since V is a Lyapunov function for $x^+ = (A + BK)x$ the sublevel set $\mathbb{X}_F = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is forward invariant for $c > 0$
- If c is selected such that $Kx \in \mathbb{U}$ for all $x \in \mathbb{X}_F$, then 0 is locally asymptotically stable and the basin of attraction contains $\mathbb{X}_F$

Constrained Optimal Control for Linear Systems

- Consider
$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$
- Cost functional
$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k)$$

Now:

- We restrict the input space to
$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m \,|\, u(k) \in \mathbb{U} \;\forall k \in \mathbb{N}\},$$
for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \operatorname{int} \mathbb{U}$
- Corresponding OCP:
$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$
- Step 1: Apply the results of the unconstrained setting (i.e., $\mathbb{U} = \mathbb{R}^m$) to obtain Lyapunov function $V(x) = x^T P_F x$ and the optimal feedback law
$$\mu(x) = Kx = -\left(R + B^T PB\right)^{-1} B^T PAx.$$
- Since V is a Lyapunov function for $x^+ = (A + BK)x$ the sublevel set $\mathbb{X}_F = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is forward invariant for $c > 0$
- If c is selected such that $Kx \in \mathbb{U}$ for all $x \in \mathbb{X}_F$, then 0 is locally asymptotically stable and the basin of attraction contains $\mathbb{X}_F$
- Step 2: For all $x_0 \in \mathbb{R}^n$ assume there exists an input $u(\cdot) \in \mathcal{U}_{\mathbb{U}}$ such that $x = 0$ can be globally asymp. stabilized.

Constrained Optimal Control for Linear Systems

- Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k)$$

Now:

- We restrict the input space to

$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m |\ u(k) \in \mathbb{U}\ \forall k \in \mathbb{N}\},$$

  for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \operatorname{int} \mathbb{U}$

- Corresponding OCP:

$$V(x_0) = \min_{u(\cdot) \in \mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$

$$\text{subject to dynamics \& initial cond.}$$

- Step 1: Apply the results of the unconstrained setting (i.e., $\mathbb{U} = \mathbb{R}^m$) to obtain Lyapunov function $V(x) = x^T P_F x$ and the optimal feedback law

$$\mu(x) = Kx = -\left(R + B^T PB\right)^{-1} B^T PAx.$$

- Since V is a Lyapunov function for $x^+ = (A+BK)x$ the sublevel set $\mathbb{X}_F = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is forward invariant for $c > 0$
- If c is selected such that $Kx \in \mathbb{U}$ for all $x \in \mathbb{X}_F$, then 0 is locally asymptotically stable and the basin of attraction contains $\mathbb{X}_F$
- Step 2: For all $x_0 \in \mathbb{R}^n$ assume there exists an input $u(\cdot) \in \mathcal{U}_{\mathbb{U}}$ such that $x = 0$ can be globally asymp. stabilized.
- Then $\forall x_0 \in \mathbb{R}^n\ \exists\ N \in \mathbb{N}$ such that $x^\star(k) \in \mathbb{X}_F\ \forall\ k \geq N$.

Constrained Optimal Control for Linear Systems

- Consider
$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$
- Cost functional
$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k)$$

Now:

- We restrict the input space to
$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m \,|\, u(k) \in \mathbb{U}\ \forall k \in \mathbb{N}\},$$
for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \operatorname{int} \mathbb{U}$
- Corresponding OCP:
$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$
$$\text{subject to dynamics \& initial cond.}$$
- Step 1: Apply the results of the unconstrained setting (i.e., $\mathbb{U} = \mathbb{R}^m$) to obtain Lyapunov function $V(x) = x^T P_F x$ and the optimal feedback law
$$\mu(x) = Kx = -\left(R + B^T PB\right)^{-1} B^T PAx.$$
- Since V is a Lyapunov function for $x^+ = (A + BK)x$ the sublevel set $\mathbb{X}_F = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is forward invariant for $c > 0$
- If c is selected such that $Kx \in \mathbb{U}$ for all $x \in \mathbb{X}_F$, then 0 is locally asymptotically stable and the basin of attraction contains $\mathbb{X}_F$
- Step 2: For all $x_0 \in \mathbb{R}^n$ assume there exists an input $u(\cdot) \in \mathcal{U}_{\mathbb{U}}$ such that $x = 0$ can be globally asymp. stabilized.
- Then $\forall x_0 \in \mathbb{R}^n\ \exists\ N \in \mathbb{N}$ such that $x^\star(k) \in \mathbb{X}_F\ \forall\ k \geq N$.
- Under the assumption that $x^\star(N) \in \mathbb{X}_F$ it holds that
$$\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} (x^\star)^T Qx^\star + (u^\star)^T Ru^\star$$
$$= \sum_{k=0}^{N-1} (x^\star(k))^T Qx^\star(k) + (u^\star(k))^T Ru^\star(k)$$
$$+ \sum_{k=N}^{\infty} (x^\star(k))^T Qx^\star(k) + (u^\star(k))^T Ru^\star(k)$$
$$= \sum_{k=0}^{N-1} (x^\star)^T Qx^\star + (u^\star)^T Ru^\star + (x^\star(N))^T P_F x^\star(N)$$
- Moreover $V(x^\star(N)) = (x^\star(N))^T P_F x^\star(N)$

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) |\ u(\cdot) \in \mathcal{U}\}$$

$$J_N(x_0, u_N(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) |\ u(\cdot) \in \mathcal{U}\}$$

$$J_N(x_0, u_N(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

- Continue to rewrite the OCP:

$$\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$

$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} \sum_{k=0}^{N-1} x^T Q x + u^T R u + x(N)^T P_F x(N)$$

$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$

$$V(x_0) = \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$

subject to dynamics & init. cond.

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) |\ u(\cdot) \in \mathcal{U}\}$$

$$J_N(x_0, u_N(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

- Continue to rewrite the OCP:

$$\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$

$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} \sum_{k=0}^{N-1} x^T Q x + u^T R u + x(N)^T P_F x(N)$$

$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$

$$V(x_0) = \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$

subject to dynamics & init. cond.

⇝ We have rewritten the infinite dimensional problem as a finite dimensional optimization problem

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) |\; u(\cdot) \in \mathcal{U}\}$$
$$J_N(x_0, u_N(\cdot)) = \textstyle\sum_{k=0}^{N-1} \ell(x(k), u(k))$$

- Continue to rewrite the OCP:

$$\min_{u(\cdot) \in \mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$
$$= \min_{u_N(\cdot) \in \mathcal{U}_{\mathbb{U}}^N} \textstyle\sum_{k=0}^{N-1} x^T Q x + u^T R u + x(N)^T P_F x(N)$$
$$= \min_{u_N(\cdot) \in \mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$

$$V(x_0) = \min_{u_N(\cdot) \in \mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond.}$$

⇝ We have rewritten the infinite dimensional problem as a finite dimensional optimization problem

- The optimal open loop input is given by

$$u^\star(\cdot) = (u_N^\star(0), \ldots, u_N^\star(N-1), Kx^\star(N), Kx^\star(N+1), \ldots)$$

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) | \; u(\cdot) \in \mathcal{U}\}$$
$$J_N(x_0, u_N(\cdot)) = \textstyle\sum_{k=0}^{N-1} \ell(x(k), u(k))$$

- Continue to rewrite the OCP:

$$\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$
$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} \textstyle\sum_{k=0}^{N-1} x^T Q x + u^T R u + x(N)^T P_F x(N)$$
$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$

$$V(x_0) = \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond.}$$

⇝ We have rewritten the infinite dimensional problem as a finite dimensional optimization problem

- The optimal open loop input is given by

$$u^\star(\cdot) = (u_N^\star(0), \ldots, u_N^\star(N-1), Kx^\star(N), Kx^\star(N+1), \ldots)$$

Note that:

- $u^\star$ and $V(x)$ are only implicitly known as the solution of the optimization problem.
- The equivalence of the OCPs needs to be understood with caution! It relies on the nontrivial assumption that $x^\star(N) \in \mathbb{X}_F$

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) |\ u(\cdot) \in \mathcal{U}\}$$
$$J_N(x_0, u_N(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

- Continue to rewrite the OCP:

$$\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$
$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} \sum_{k=0}^{N-1} x^T Q x + u^T R u + x(N)^T P_F x(N)$$
$$= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$

$$V(x_0) = \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond.}$$

⇝ We have rewritten the infinite dimensional problem as a finite dimensional optimization problem

- The optimal open loop input is given by

$$u^\star(\cdot) = (u_N^\star(0), \ldots, u_N^\star(N-1), Kx^\star(N), Kx^\star(N+1), \ldots)$$

Note that:

- $u^\star$ and $V(x)$ are only implicitly known as the solution of the optimization problem.
- The equivalence of the OCPs needs to be understood with caution! It relies on the nontrivial assumption that $x^\star(N) \in \mathbb{X}_F$
- Alternatively, consider *terminal constraints*

$$\min_{u^N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond., } x(N) \in \mathbb{X}_F$$

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) | \, u(\cdot) \in \mathcal{U}\}$$
$$J_N(x_0, u_N(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

- Continue to rewrite the OCP:

$$\begin{aligned}
&\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot)) \\
&= \min_{u_N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} \sum_{k=0}^{N-1} x^T Q x + u^T R u + x(N)^T P_F x(N) \\
&= \min_{u_N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)
\end{aligned}$$

$$V(x_0) = \min_{u_N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond.}$$

⇝ We have rewritten the infinite dimensional problem as a finite dimensional optimization problem

- The optimal open loop input is given by

$$u^\star(\cdot) = (u_N^\star(0), \ldots, u_N^\star(N-1), Kx^\star(N), Kx^\star(N+1), \ldots)$$

Note that:

- $u^\star$ and $V(x)$ are only implicitly known as the solution of the optimization problem.
- The equivalence of the OCPs needs to be understood with caution! It relies on the nontrivial assumption that $x^\star(N) \in \mathbb{X}_F$
- Alternatively, consider *terminal constraints*

$$\min_{u^N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond.,} \quad x(N) \in \mathbb{X}_F$$

- However
  - in this case the optimal solution might not be optimal with respect to the cost function (i.e., it might be cheaper to reach $\mathbb{X}_F$ in more than N steps)
  - the optimization problem is infeasible if it is not possible to reach the set $\mathbb{X}_F$ in N steps

Constrained Optimal Control for Linear Systems

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \dots, u(N-1)) |\ u(\cdot) \in \mathcal{U}\}$$
$$J_N(x_0, u_N(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

- Continue to rewrite the OCP:

$$\begin{aligned} &\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot)) \\ &= \min_{u_N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} \sum_{k=0}^{N-1} x^T Q x + u^T R u + x(N)^T P_F x(N) \\ &= \min_{u_N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N) \end{aligned}$$

$$V(x_0) = \min_{u_N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond.}$$

⇝ We have rewritten the infinite dimensional problem as a finite dimensional optimization problem

- The optimal open loop input is given by

$$u^\star(\cdot) = (u_N^\star(0), \dots, u_N^\star(N-1), Kx^\star(N), Kx^\star(N+1), \dots)$$

Note that:

- $u^\star$ and $V(x)$ are only implicitly known as the solution of the optimization problem.
- The equivalence of the OCPs needs to be understood with caution! It relies on the nontrivial assumption that $x^\star(N) \in \mathbb{X}_F$
- Alternatively, consider *terminal constraints*

$$\min_{u^N(\cdot)\in\mathcal{U}^N_{\mathbb{U}}} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$
$$\text{subject to dynamics \& init. cond.,}\ x(N) \in \mathbb{X}_F$$

- However
  - in this case the optimal solution might not be optimal with respect to the cost function (i.e., it might be cheaper to reach $\mathbb{X}_F$ in more than N steps)
  - the optimization problem is infeasible if it is not possible to reach the set $\mathbb{X}_F$ in N steps

⇝ These ideas are extended in MPC (in the next chapter)!

Introduction to Nonlinear Control: Stability, control design, and estimation