Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

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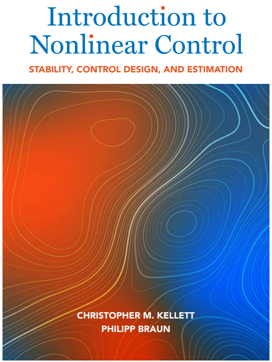

Introduction to Nonlinear Control: Stability, control design, and estimation

Part III: Observer Design and Estimation

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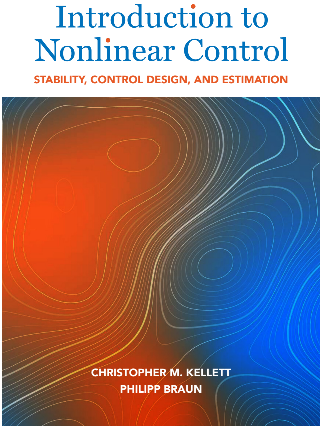

Classical Observer Design

So far:

- The concepts so far rely on the knowledge of the state $x \in \mathbb{R}^n$.
- The full state x is in general not known and only the output $y \in \mathbb{R}^p$ is available.

$\rightsquigarrow$ A controller design can not, in general, rely on the full state x.

$\rightsquigarrow$ An estimate $\hat{x}$ of the state needs to be derived (observability, detectibility)

- If $\hat{x}(t) \to x(t)$ for $t \to \infty$, can $\hat{x}$ be used for the definition of a feedback controller $u(\hat{x})$?

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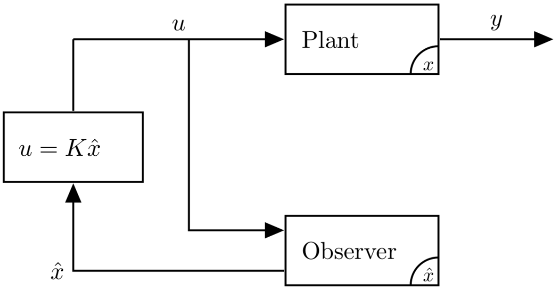

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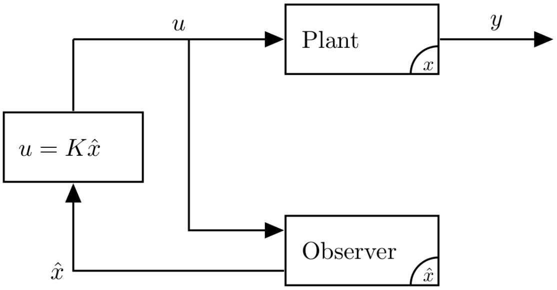

Consider Linear systems:

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\begin{aligned}
\dot{x} &= Ax + Bu, \qquad x(0) \in \mathbb{R}^n, \\
y &= Cx + Du
\end{aligned}
$$

- We assume that $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are known, while the internal state $x \in \mathbb{R}^n$ and the initial condition $x(0)$ are unknown.
- Assume that the matrix A is Hurwitz.

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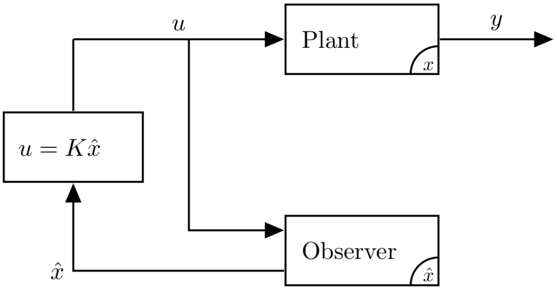

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- Introduce observer dynamics as a copy of the system

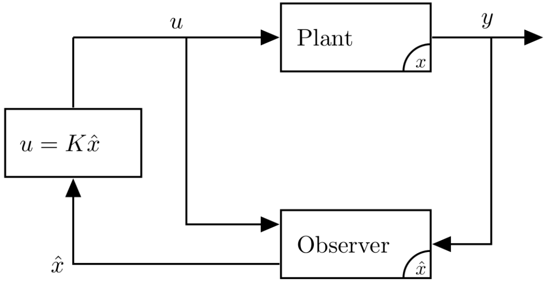
$$\dot{\hat{x}} = A\hat{x} + Bu, \qquad \hat{x}(0) \in \mathbb{R}^n$$

  - $\hat{x} \in \mathbb{R}^n$ estimate of the state $x \in \mathbb{R}^n$
  - Estimation error $e = x - \hat{x}$

- Error dynamics:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu = A(x - \hat{x}) = Ae$$
$$\hat{x}(t) \to x(t) \quad \Leftrightarrow \quad e(t) \to 0 \quad \Leftrightarrow A \text{ Hurwitz}$$

Luenberger Observers



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$$\dot{x} = Ax + Bu, \qquad x(0) \in \mathbb{R}^n, \qquad (\text{or } x^+ = Ax + Bu)$$
$$y = Cx + Du$$

Define observer dynamics:

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - \hat{y}),$$
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- *output injection term* $L \in \mathbb{R}^{n \times p}$

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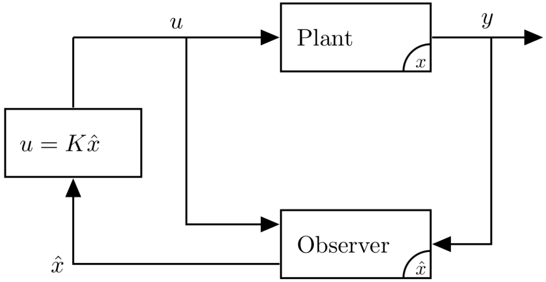

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$\rightsquigarrow$ The error dynamics are independent of u

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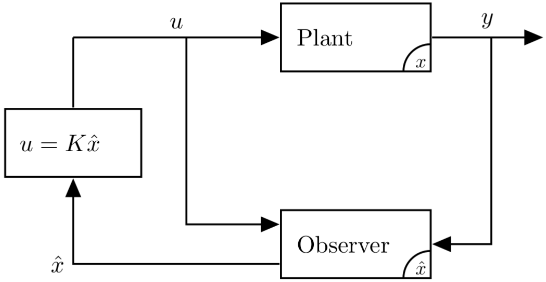

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- $A + LC$ has the same eigenvalues as $(A + LC)^T = A^T + C^T L^T$

⇝ If (A, C) is observable, the poles of $A + LC$ can be placed arbitrarily, i.e., L can be defined such that $A + LC$ is Hurwitz.

⇝ If (A, C) is detectable, then there exists L such that $A + LC$ is Hurwitz.

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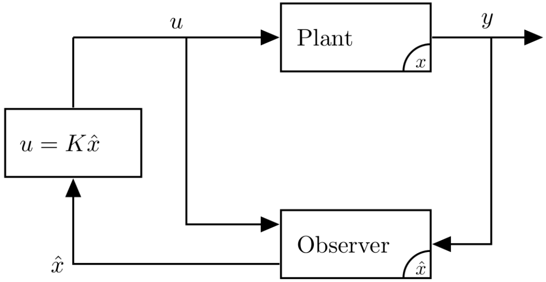

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- See pole placement
- x can be approximated through $\hat{x}$

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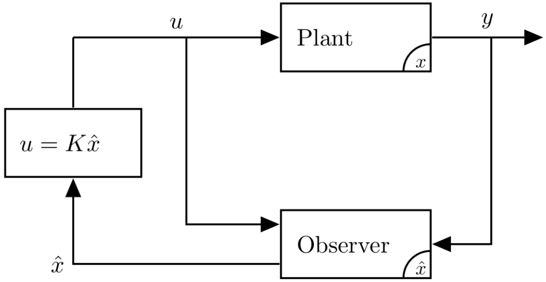

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- x can be approximated through $\hat{x}$
- Controller design $u = K\hat{x}$?

Luenberger Observers & Controller design

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$$u(\hat{x}) = K\hat{x}, \qquad K \in \mathbb{R}^{m \times n}$$

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Overall closed loop system

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

- If (A, B) is controllable and (A, C) is observable, we can place the poles of the closed-loop system arbitrarily by choosing K and L.
- The convergence $|x(t)| \to 0$ and $|e(t)| \to 0$ for $t \to \infty$ can be guaranteed by designing L and K individually. $\rightsquigarrow$ *separation principle*
- (The separation principle is only true for the asymptotic behavior)

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Alternative representation in terms of x and $\hat{x}$:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$\rightsquigarrow$ While the separation principle is not visible the dynamics capture the same information.

Minimum Energy Estimator

Recall

- We have discussed optimal control for linear systems
- Can we derive an optimal estimator (in terms of minimal energy)?

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The *minimum energy estimation* problem:

- For given $u(\cdot)$, $y(\cdot)$, find $\bar{x} : \mathbb{R}_{\leq t_0} \to \mathbb{R}^n$ for $t_0 \geq 0$, which satisfies the dynamics

$$\begin{aligned}\dot{\bar{x}} &= A\bar{x} + Bu + \bar{B}v \\ y &= C\bar{x} + Du + w\end{aligned}$$

an which minimizes the cost function

$$J_{\mathrm{MEE}}(\bar{x}(t_0), v(\cdot)) = \int_{-\infty}^{t_0} w(\tau)^T Q w(\tau) + v(\tau)^T R v(\tau) d\tau$$

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Note that:

- Design parameters: $Q \in \mathcal{S}^p_{>0}$, $R \in \mathcal{S}^q_{>0}$
- $J_{\mathrm{MEE}}(\bar{x}(t_0), v(\cdot))$ is a function of $v(\cdot)$ but not $w(\cdot)$:

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- Q large: penalize noise $w(\cdot)$; neglect disturbance
- R large: penalize disturbance $v(\cdot)$; neglect noise

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Theorem (The minimum energy estimator)

- *Consider the perturbed linear system and assume that $(A, \bar{B})$ is controllable and (A, C) is detectable.*
- *Consider the optimization problem where the cost function is defined through positive definite matrices $Q \in \mathcal{S}^p_{>0}$ and $R \in \mathcal{S}^q_{>0}$.*
- *Then there exists $S \in \mathcal{S}^n_{>0}$ to the dual algebraic Riccati equation*

$$AS + SA^T + \bar{B}R^{-1}\bar{B}^T - SC^TQCS = 0$$

such that $A - LC$ is Hurwitz, where $L = SC^TQ$.

- *The minimum energy estimator is given by*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du)$$

and the initial condition $\hat{x}(t_0) = \bar{x}_0$, $t_0 \in \mathbb{R}_{\geq 0}$.

Minimum Energy Estimator (and the Kalman Filter)

Remarks:

- The *minimum energy estimator* is derived using the *deterministic setting*
- In the *stochastic setting* the minimum energy estimator is related to the (cont. time) Kalman filter.
- Consider

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- Assume $v(\cdot)$ and $w(\cdot)$ represent functions of zero-mean Gaussian white noise with covariance matrices satisfying

$$\begin{aligned} \mathrm{E}[v(t)v(\tau)^T] &= \delta(t-\tau)R^{-1}, \\ \mathrm{E}[w(t)w(\tau)^T] &= \delta(t-\tau)Q^{-1}, \\ \mathrm{E}[v(t)w(\tau)^T] &= 0 \end{aligned}$$

for all $t, \tau \in \mathbb{R}$ and $Q > 0$, $R > 0$.

Here:

- *expected value*: $\mathrm{E}[\cdot]$:
- *Dirac delta function*: $\delta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

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Under these conditions

- $\hat{x}$ obtained through the minimum energy estimator minimizes the expected value

$$\lim_{t\to\infty} \mathrm{E}\left[|x(t) - \hat{x}(t)|^2\right]$$

(which is used to derive the Kalman filter)

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- The *minimum energy estimator* is derived using the *deterministic setting*
- In the *stochastic setting* the minimum energy estimator is related to the (cont. time) Kalman filter.
- Consider

$$\dot{x} = Ax + Bu + \bar{B}v$$
$$y = Cx + Du + w$$

- Assume $v(\cdot)$ and $w(\cdot)$ represent functions of zero-mean Gaussian white noise with covariance matrices satisfying

$$\mathrm{E}[v(t)v(\tau)^T] = \delta(t-\tau)R^{-1},$$
$$\mathrm{E}[w(t)w(\tau)^T] = \delta(t-\tau)Q^{-1},$$
$$\mathrm{E}[v(t)w(\tau)^T] = 0$$

for all $t, \tau \in \mathbb{R}$ and $Q > 0$, $R > 0$.

Here:

- *expected value*: $\mathrm{E}[\cdot]$:
- *Dirac delta function*: $\delta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

Under these conditions

- $\hat{x}$ obtained through the minimum energy estimator minimizes the expected value

$$\lim_{t\to\infty} \mathrm{E}\left[|x(t) - \hat{x}(t)|^2\right]$$

(which is used to derive the Kalman filter)

Similarly, the minimum energy estimator and the Kalman filter can be derived in the discrete time setting.

Introduction to Nonlinear Control: Stability, control design, and estimation

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