Introduction to Nonlinear Control

Stability, control design, and estimation

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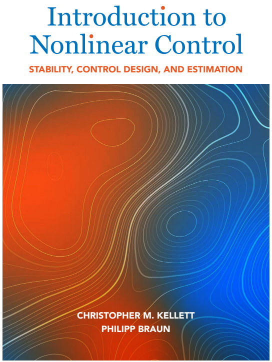
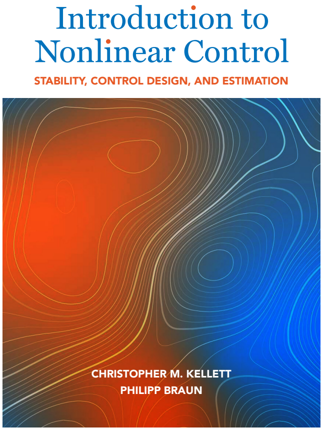


Part III: Observer Design and Estimation



Extended Kalman Filter – Continuous time setting

Consider (nonlinear) system with (nonlinear) output:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)), \qquad x(0) \in \mathbb{R}^n \\
y(t) &= h(x(t)).
\end{aligned}
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Consider observer dynamics:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + L(t)(y(t) - h(\hat{x}(t)))$$

with

- state estimate $\hat{x} \in \mathbb{R}^n$
- $L(\cdot)$ time-dependent output injection term to be designed

Observer design:

- Error $e = x - \hat{x}$ and error dynamics

$$\dot{e} = f(x, u) - f(\hat{x}, u) - L(t)(h(x) - h(\hat{x}))$$

- Define (time-varying linearization in $(\hat{x}, u)$)

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)) \quad \text{and} \quad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t))$$

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Note that: It holds that

$$f(e + \hat{x}, u) - f(\hat{x}, u) = 0 \quad \text{for} \quad e = 0$$
$$\frac{\partial}{\partial e}\left(f(x, u) - f(\hat{x}, u)\right)\Big|_{e=0} = \frac{\partial}{\partial e} f(e + \hat{x}, u)\Big|_{e=0} = A(t)$$
$$\frac{\partial}{\partial e}\left(h(x) - h(\hat{x})\right)\Big|_{e=0} = \frac{\partial}{\partial e} h(e + \hat{x})\Big|_{e=0} = C(t)$$

The Taylor approximation at $e = 0$ w.r.t. $\hat{x}$ can be written as

$$\dot{e} = (A(t) - L(t)C(t))e + \Delta(e, x, u)$$

where $\Delta(e, x, u)$ denotes the error term.

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Consider candidate Lyapunov function $V : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$

$$V(e(t)) = e(t)^T P^{-1}(t) e(t), \qquad \text{for } P > 0.$$

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$$V(e(t)) = e(t)^T P^{-1}(t) e(t), \qquad \text{for } P > 0.$$

Lemma

Consider $P : \mathbb{R} \to \mathbb{R}^{n \times n}$, positive definite, cont. diff. Then

$$\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t).$$

Extended Kalman Filter: the candidate Lyapunov function

Derivative of the candidate Lyapunov function: (Recall: $\dot{e} = (A(t) - L(t)C(t))e + \Delta(e, x, u)$)

$$\dot{V}(e) = \dot{e}^T P^{-1} e + e^T \dot{P}^{-1} e + e^T P^{-1} \dot{e} = ((A - LC)e + \Delta)^T P^{-1} e + e^T P^{-1} ((A - LC)e + \Delta) - e^T P^{-1} \dot{P} P^{-1} e$$

$$= e^T P^{-1} \left(P(A - LC)^T + (A - LC)P - \dot{P} \right) P^{-1} e + 2e^T P^{-1} \Delta$$

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$$= e^T P^{-1} \left(P(A - LC)^T + (A - LC)P - \dot{P} \right) P^{-1} e + 2e^T P^{-1} \Delta$$

Select $L(t) = P(t)C(t)^T Q$ for $Q > 0$. Then:

$$\dot{V}(e) = e^T P^{-1} \left(PA^T + AP - 2PC^T QCP - \dot{P} \right) P^{-1} e + 2e^T P^{-1} \Delta$$

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If $P(t)$ satisfies the differential Riccati equation ($P(t_0) > 0$, $R > 0$)

$$\dot{P}(t) = P(t)A(t)^T + A(t)P(t) - P(t)C(t)^T QC(t)P(t) + R^{-1}$$

then

$$\dot{V}(e) = -e^T P^{-1} \left(PC^T QCP + R^{-1} \right) P^{-1} e + 2e^T P^{-1} \Delta$$

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Note that:

- For $e \neq 0$ small, it holds that $\dot{V}(e) < 0$
- Thus, the $e = 0$ is locally exponentially stable.

(Continuous time) Extended Kalman Filter: Summary

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Observer dynamics:

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Error dynamics

$$\dot{e} = f(x, u) - f(\hat{x}, u) - L(t)(h(x) - h(\hat{x}))$$

Note that:

- For nonlinear systems, local exponential stability $\hat{x}(t) \to x(t)$ is obtained

Equations of the extended Kalman filter:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + P(t) \left(\frac{\partial h}{\partial x}(\hat{x}(t))\right)^T Q(y(t) - h(\hat{x}(t)))$$

$$\dot{P}(t) = P \left(\frac{\partial f}{\partial x}(\hat{x}(t), u(t))\right)^T + \left(\frac{\partial f}{\partial x}(\hat{x}(t), u(t))\right) P - P \left(\frac{\partial h}{\partial x}(\hat{x}(t))\right)^T Q \left(\frac{\partial h}{\partial x}(\hat{x}(t))\right) P + R^{-1}$$

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Concluding remarks

- The extended Kalman filters is based on the linearization around the current state estimate $\hat{x}$
- Higher order terms can also be considered in the observer design ($\rightsquigarrow$ unscented Kalman filter)
- Derivations can also be done in the discrete-time setting
- Constraints can be taken into account through moving horizon estimation (dual to MPC)

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