

Introduction to Nonlinear Control

Stability, control design, and estimation

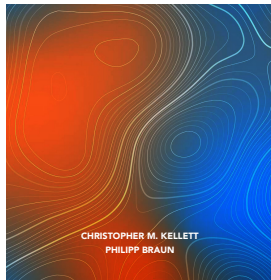
Christopher M. Kellett & Philipp Braun



Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION

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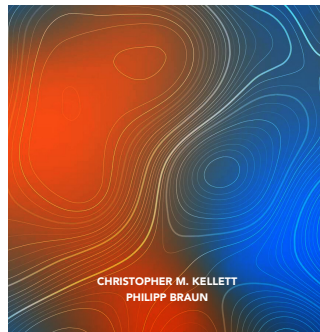
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Introduction to Nonlinear Control

STABILITY, CONTROL DESIGN, AND ESTIMATION



Extended Kalman Filter – Continuous time setting

Consider (nonlinear) system with (nonlinear) output:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), & x(0) &\in \mathbb{R}^n \\ y(t) &= h(x(t)).\end{aligned}$$

Extended Kalman Filter – Continuous time setting

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Consider observer dynamics:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + L(t)(y(t) - h(\hat{x}(t)))$$

with

- state estimate $\hat{x} \in \mathbb{R}^n$
- $L(\cdot)$ time-dependent output injection term to be designed

Observer design:

- **Error** $e = x - \hat{x}$ and **error dynamics**

$$\dot{e} = f(x, u) - f(\hat{x}, u) - L(t)(h(x) - h(\hat{x}))$$

- **Define** (time-varying linearization in $(\hat{x}, u)$)

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)) \quad \text{and} \quad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t))$$

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Note that: It holds that

$$\begin{aligned}f(e + \hat{x}, u) - f(\hat{x}, u) &= 0 \quad \text{for } e = 0 \\ \frac{\partial}{\partial e} (f(x, u) - f(\hat{x}, u)) \Big|_{e=0} &= \frac{\partial}{\partial e} f(e + \hat{x}, u) \Big|_{e=0} = A(t) \\ \frac{\partial}{\partial e} (h(x) - h(\hat{x})) \Big|_{e=0} &= \frac{\partial}{\partial e} h(e + \hat{x}) \Big|_{e=0} = C(t)\end{aligned}$$

The Taylor approximation at $e = 0$ w.r.t. $\hat{x}$ can be written as

$$\dot{e} = (A(t) - L(t)C(t))e + \Delta(e, x, u)$$

where $\Delta(e, x, u)$ denotes the error term.

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Consider candidate Lyapunov function $V : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}_{\geq 0}$

$$V(e(t)) = e(t)^T P^{-1}(t) e(t), \quad \text{for } P > 0.$$

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$$V(e(t)) = e(t)^T P^{-1}(t) e(t), \quad \text{for } P > 0.$$

Lemma

Consider $P : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, positive definite, cont. diff. Then

$$\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t).$$

Extended Kalman Filter: the candidate Lyapunov function

Derivative of the candidate Lyapunov function: (Recall: $\dot{e} = (A(t) - L(t)C(t))e + \Delta(e, x, u)$)

$$\begin{aligned}\dot{V}(e) &= \dot{e}^T P^{-1} e + e^T \dot{P}^{-1} e + e^T P^{-1} \dot{e} = ((A - LC)e + \Delta)^T P^{-1} e + e^T P^{-1} ((A - LC)e + \Delta) - e^T P^{-1} \dot{P} P^{-1} e \\ &= e^T P^{-1} \left(P(A - LC)^T + (A - LC)P - \dot{P} \right) P^{-1} e + 2e^T P^{-1} \Delta\end{aligned}$$

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Select $L(t) = P(t)C(t)^T Q$ for $Q > 0$. Then:

$$\dot{V}(e) = e^T P^{-1} \left(PA^T + AP - 2PC^T QCP - \dot{P} \right) P^{-1} e + 2e^T P^{-1} \Delta$$

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If $P(t)$ satisfies the differential Riccati equation ($P(t_0) > 0, R > 0$)

$$\dot{P}(t) = P(t)A(t)^T + A(t)P(t) - P(t)C(t)^T QC(t)P(t) + R^{-1}$$

then

$$\dot{V}(e) = -e^T P^{-1} \left(PC^T QCP + R^{-1} \right) P^{-1} e + 2e^T P^{-1} \Delta$$

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Note that:

- For $e \neq 0$ small, it holds that $\dot{V}(e) < 0$
- Thus, the $e = 0$ is locally exponentially stable.

(Continuous time) Extended Kalman Filter: Summary

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Note that:

- For nonlinear systems, local exponential stability $\hat{x}(t) \rightarrow x(t)$ is obtained

Equations of the extended Kalman filter:

$$\begin{aligned}\dot{\hat{x}}(t) &= f(\hat{x}(t), u(t)) + P(t) \left(\frac{\partial h}{\partial x}(\hat{x}(t)) \right)^T Q (y(t) - h(\hat{x}(t))) \\ \dot{P}(t) &= P \left(\frac{\partial f}{\partial x}(\hat{x}(t), u(t)) \right)^T + \left(\frac{\partial f}{\partial x}(\hat{x}(t), u(t)) \right) P - P \left(\frac{\partial h}{\partial x}(\hat{x}(t)) \right)^T Q \left(\frac{\partial h}{\partial x}(\hat{x}(t)) \right) P + R^{-1}\end{aligned}$$

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Concluding remarks

- The extended Kalman filter is based on the linearization around the current state estimate $\hat{x}$
- Higher order terms can also be considered in the observer design ($\leadsto$ unscented Kalman filter)
- Derivations can also be done in the discrete-time setting
- Constraints can be taken into account through moving horizon estimation (dual to MPC)

Introduction to Nonlinear Control: Stability, control design, and estimation

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5. Discrete Time Systems
6. Absolute Stability
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