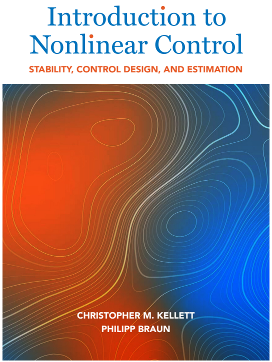
Introduction to Nonlinear Control

Stability, control design, and estimation

Christopher M. Kellett & Philipp Braun

PRINCETON UNIVERSITY PRESS

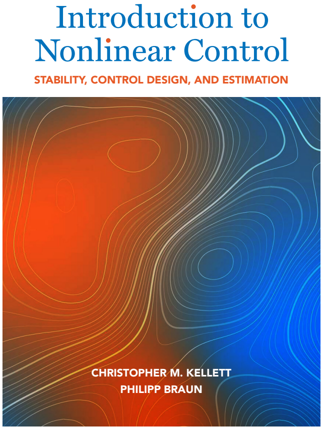


Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

6 Absolute Stability

- 6.1 A Commonly Ignored Design Issue
- 6.2 Historical Perspective on the Lur'e Problem
- 6.3 Sufficient Conditions for Absolute Stability
  - 6.3.1 Circle Criterion
  - 6.3.2 Popov Criterion
  - 6.3.3 Circle versus Popov Criterion
- 6.4 Exercises
- 6.5 Bibliographical Notes and Further Reading



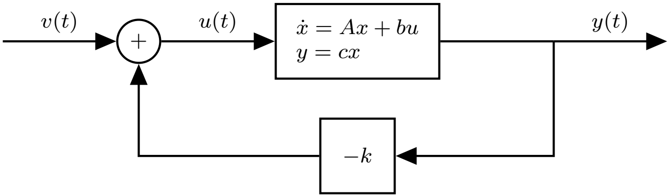
A Commonly Ignored Design Issue

Linear system: $(A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^{n\times 1}, c \in \mathbb{R}^{1\times n})$

$$\dot{x} = Ax + bu, \qquad y = cx,$$

Feedback interconnection: $u = -ky$

$$\dot{x} = (A - bkc)x,$$



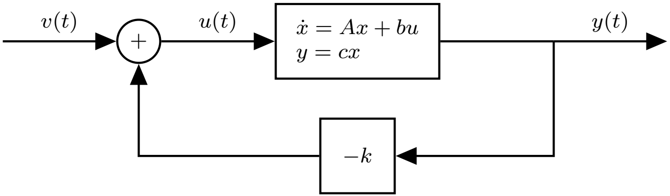
A Commonly Ignored Design Issue

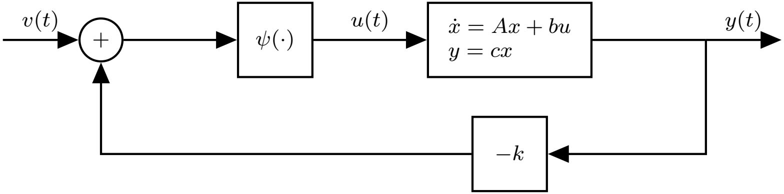
Linear system: $(A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^{n\times 1}, c \in \mathbb{R}^{1\times n})$

$$\dot{x} = Ax + bu, \qquad y = cx,$$

Feedback interconnection: $u = -ky$

$$\dot{x} = (A - bkc)x,$$





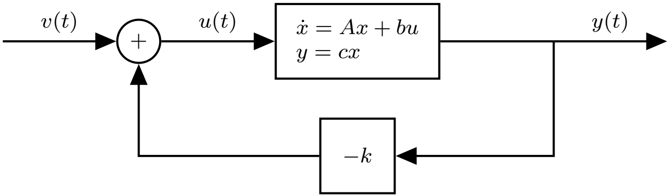
A Commonly Ignored Design Issue

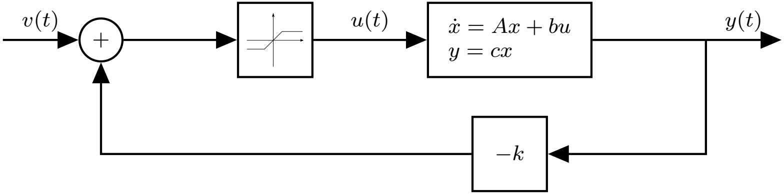
Linear system: ($A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, $c \in \mathbb{R}^{1\times n}$)

$$\dot{x} = Ax + bu, \qquad y = cx,$$

Feedback interconnection: $u = -ky$

$$\dot{x} = (A - bkc)x,$$



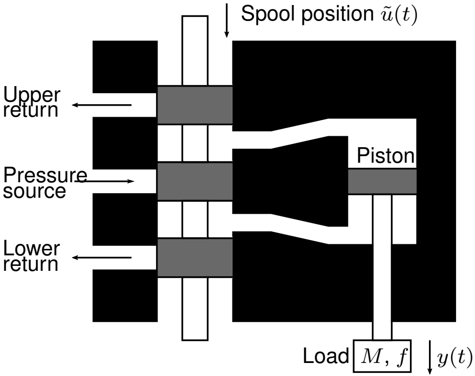


Saturation function

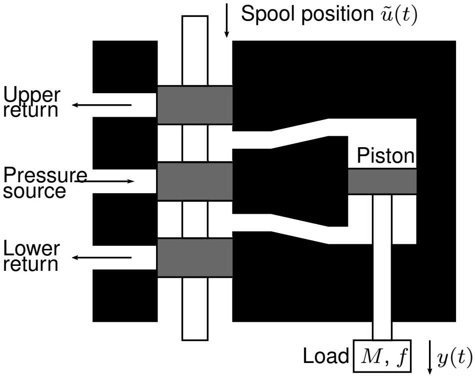
$\mathrm{sat} : \mathbb{R} \to [-1, 1]$:

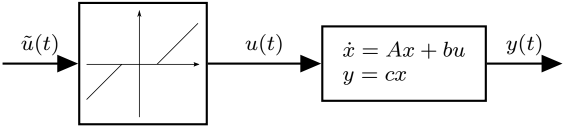
$$\mathrm{sat}(z) = \begin{cases} -1, & \text{for } z \le -1 \\ z, & \text{for } z \in [-1, 1] \\ 1, & \text{for } z \ge 1 \end{cases}$$

A Commonly Ignored Design Issue (Example: A servo-valve)



A Commonly Ignored Design Issue (Example: A servo-valve)





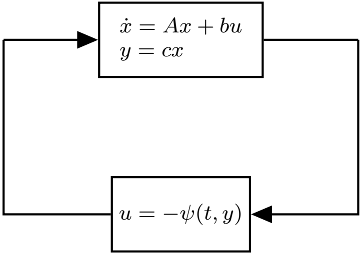
Deadzone function

$\mathrm{dz} : \mathbb{R} \to \mathbb{R}$:

$$\mathrm{dz}(z) = \begin{cases} z+1, & \text{for } z \leq -1 \\ 0, & \text{for } z \in [-1, 1] \\ y-1, & \text{for } z \geq 1 \end{cases}$$

A Commonly Ignored Design Issue (The Lur'e Problem)

Consider the feedback interconnection:

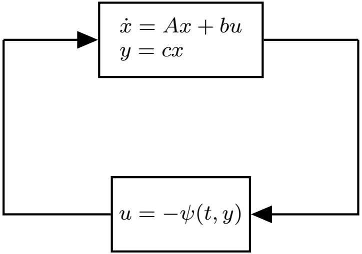


Lur'e problem:

- Which conditions on the functions $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ guarantee asymptotic stability of the origin?

A Commonly Ignored Design Issue (The Lur'e Problem)

Consider the feedback interconnection:

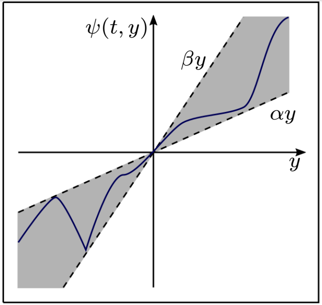


Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

for all $t \geq 0$ and for all $y \in \Omega$. For $\Omega = \mathbb{R}$ we say that the sector condition is satisfied globally.



Lur'e problem:

- Which conditions on the functions $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ guarantee asymptotic stability of the origin?

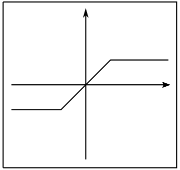
A Commonly Ignored Design Issue (The Sector Condition)

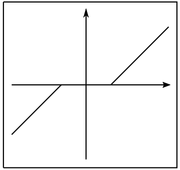
Common nonlinearities: $\operatorname{sign}: \mathbb{R} \to \mathbb{R}$,

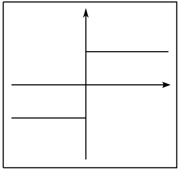
$$\operatorname{sat}(y) = \begin{cases} -1, & \text{for } y \leq -1, \\ y, & \text{for } -1 \leq y \leq 1, \\ 1, & \text{for } y \geq 1. \end{cases}$$

$$\operatorname{dz}(y) = \begin{cases} y+1, & \text{for } y \leq -1, \\ 0, & \text{for } -1 \leq y \leq 1, \\ y-1, & \text{for } y \geq 1. \end{cases}$$

$$\operatorname{sign}(y) = \begin{cases} -1, & \text{for } y < 0, \\ 0, & \text{for } y = 0, \\ 1, & \text{for } y > 0, \end{cases}$$







Question:

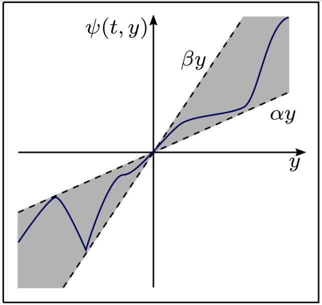
- Which nonlinearity satisfies a sector condition?

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

for all $t \geq 0$ and for all $y \in \Omega$. For $\Omega = \mathbb{R}$ we say that the sector condition is satisfied globally.



Absolute Stability

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

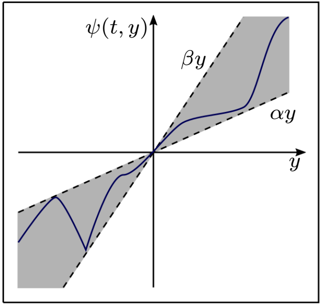
for all $t \geq 0$ and for all $y \in \Omega$. For $\Omega = \mathbb{R}$ we say that the sector condition is satisfied globally.

Definition (Absolute stability)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. The Lur'e system

$$\dot{x} = Ax - b\psi(t, y)$$

is called absolutely stable (with respect to α, β, Ω) if the origin is asymptotically stable for all $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfying the sector condition for all $t \geq 0$ and for all $y_0 \in \Omega$.



Historical Perspective on the Lur'e Problem

Conjecture (Aizerman's Conjecture (1949))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system

$$
\begin{aligned}
\dot{x} &= Ax + bu, \\
y &= cx
\end{aligned}
$$

is globally asymptotically stable for all linear feedback

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedback in the sector

$$\alpha \leq \frac{\psi(y)}{y} \leq \beta, \quad y \neq 0.$$

$\rightsquigarrow$ Conjecture was shown to be wrong through counterexamples.

Historical Perspective on the Lur'e Problem

Conjecture (Aizerman's Conjecture (1949))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system

$$\dot{x} = Ax + bu,$$
$$y = cx$$

is globally asymptotically stable for all linear feedback

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedback in the sector

$$\alpha \leq \frac{\psi(y)}{y} \leq \beta, \quad y \neq 0.$$

$\rightsquigarrow$ Conjecture was shown to be wrong through counterexamples.

Conjecture (Kalman's Conjecture (1957))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system

$$\dot{x} = Ax + bu,$$
$$y = cx$$

is globally asymptotically stable for all linear feedback

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedback belonging to the incremental sector

$$\alpha \leq \tfrac{\partial}{\partial y}\psi(y) \leq \beta.$$

$\rightsquigarrow$ Conjecture was shown to be wrong through counterexamples.

Absolute stability through the Circle Criterion and Popov Criterion

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2 \qquad \forall\, y \in \mathbb{R}, \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Then the system is absolutely stable if:

1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

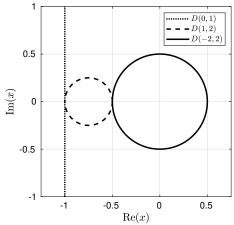
Definitions: (Disc in the complex plane)

- center $\sigma : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- radius $r : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- for $\alpha \neq 0$ and $\beta > 0$ we define

$$\sigma(\alpha, \beta) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right), \quad r(\alpha, \beta) = \frac{\mathrm{sign}(\alpha)}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$$

Then, the disc $D(\cdot, \cdot)$ is defined as

$$D(\alpha, \beta) = \begin{cases} \{x \in \mathbb{C} : x = -\frac{1}{\beta} + j\omega, \omega \in \mathbb{R}\}, & \alpha = 0 < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha, \beta)| = r(\alpha, \beta)\}, & 0 < \alpha < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha, \beta)| = r(\alpha, \beta)\}, & \alpha < 0 < \beta. \end{cases}$$



Absolute stability through the Circle Criterion and Popov Criterion

Theorem (Circle Criterion)

Suppose (A, b, c) is a minimal realization of $G(s)$ and $\psi(t, y)$ satisfies the sector condition

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2 \qquad \forall\, y \in \mathbb{R}, \quad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Then the system is absolutely stable if:

1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of $D(0, \beta)$) and $G(s)$ is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk $D(\alpha, \beta)$, and encircles it in the counter-clockwise direction as many times, N, as there are right-half plane poles of $G(s)$; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk $D(\alpha, \beta)$, and $G(s)$ is Hurwitz.*

Theorem (Popov Criterion)

Suppose A is Hurwitz, (A, b) is controllable, (A, c) is observable, and $\psi(y)$ satisfies the sector condition

$$0 \leq y\psi(y) \leq \beta y^2 \qquad \forall\, y \in \mathbb{R}.$$

Then the Lur'e system with

$$G(s) = c(sI - A)^{-1}b$$

is absolutely stable if there is an $\eta \geq 0$ with $-\frac{1}{\eta}$ not an eigenvalue of A such that

$$H(s) = 1 + (1 + \eta s)\beta G(s)$$

is strictly positive real.

Introduction to Nonlinear Control: Stability, control design, and estimation

Part I: Dynamical Systems

1. Nonlinear Systems - Fundamentals & Examples
2. Nonlinear Systems - Stability Notions
3. Linear Systems and Linearization
4. Frequency Domain Analysis
5. Discrete Time Systems
6. Absolute Stability
7. Input-to-State Stability

Part II: Controller Design

8. LMI Based Controller and Antiwindup Designs
9. Control Lyapunov Functions
10. Sliding Mode Control
11. Adaptive Control
12. Introduction to Differential Geometric Methods
13. Output Regulation
14. Optimal Control
15. Model Predictive Control

Part III: Observer Design & Estimation

16. Observer Design for Linear Systems
17. Extended & Unscented Kalman Filter & Moving Horizon Estimation
18. Observer Design for Nonlinear Systems